Research Statement

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My primary research area is low dimensional topological dynamics, especially the theory of non-abelian group actions on the circle. I have also made contributions to the theory of formal languages, with an aim to solve combinatorial group theory problems using topological methods. I am broadly interested in geometric group theory and applied nonlinear dynamical systems and chaos theory related topics as well.

EXTREMAL ACTIONS OF FREE GROUPS ON THE CIRCLE (§1) . The first research program can be roughly summarized as: *Given only the information about some dynamical invariant of a number of periodic processes, what information can we predict about the asymptotic behavior of their **composition** in some specific order.* We formulate the periodic processes below in terms of free group action on a circle, and the composition as the action of a word in the group.

The Poincaré rotation number, denoted $\widetilde{\mathrm{rot}}$, is a real-valued semi-conjugacy invariant function on $\widetilde{\mathrm{Homeo}}_+(S^1)$, the universal central extension of $\mathrm{Homeo}_+(S^1)$, the group of orientation-preserving homeomorphisms of the circle. If w is a word in the free group $F := \langle a.b \rangle$, given real numbers $r = \widetilde{\mathrm{rot}}(\rho(a)), s = \widetilde{\mathrm{rot}}(\rho(b))$, we would like to understand the set $X(w; r, s) := \{\widetilde{\mathrm{rot}}(\rho(w))\}$ and in particular, $R(w; r, s) := \sup_\rho X(w; r, s)$, taken over all $\rho : F \to \widetilde{\mathrm{Homeo}}_+(S^1)$.

If w is in the semigroup generated by a and b, plotting the graph of R against r and s gives a staircase like structure (see fig. 1a, [3]), called a Ziggurat. When $r = p/q$ is rational, it turns out that $R(w; p/q, t) = h_a(w)p/q + h_b(w)$ on an entire interval $t \in [1 - \mathrm{fr}_w(p/q), 1)$, where h_a and h_b count the number a's and b's in w, and $\mathrm{fr}_w(p/q)$ is a rational number, a priori depending on p/q in a complicated way. In [6], I proved that $\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q}$, where $g = gcd(q, h_b(w))$ and $\sigma_w(g) \cdot g \in \mathbb{Z}$ (with an algorithm for calculating it in $\mathcal{O}(1)$ time). This shows fr_w satisfies a *power law* and the maximal intervals are analogues of Arnol'd tongues in this context (see §1.1). This formula also explains the periodic nature of the ziggurat near the slopes (see fig. 1b).

Future plans and opportunities for student research. As my ongoing and future work in this project, I have plan to prove further rigidity and stability properties of this graph. For example, I have proved that the slippery conjecture 1.5 holds in some specific cases. In immediate future, I wish to understand more about the 'position' of the stairs in the Ziggurat and prove further bounds on the value of R. The opportunity for student involvement in this program comes from the fact that using some dynamical and number theoretic arguments we can reformulate the problem as a linear programming problem, which makes the tools and techniques more accessible to student researchers.

Additionally, the analogous problem for the general case of semi-positive and arbitrary words (see §1.2) is still unexplored. Because of the nature of the problem, it is hard to formulate the process using a discrete model. I have proved and conjectured similar rationality results in this area using ideas from one dimensional dynamics and theory of Diophantine approximations. I expect that further progress in this are will use tools from Analysis and Nonlinear Dynamical Systems.

A TOPOLOGICAL APPROACH TO PROVING RESULTS IN FORMAL LANGUAGE THEORY (§2) . As the title suggests, this research program can be roughly summarized as: *Given a group theoretical formulation of certain hierarchical properties in mathematical linguistics, can we use topological tools such as homology and degree to answer whether a given word is trivial or not.*

A finitely generated group $G = \langle \Sigma \mid R \rangle$ is called context-free (CF) if its word problem $\{w \in \Sigma^\star \mid w =_R 1\}$ has a presentation where a set of replacement rules produce all possible strings in it via recursion.

Such hierarchical properties in mathematical linguistics often relate to combinatorial and algebraic properties of groups and their Cayley graph in significant ways. Building on works of Salvati [16] and Nederhof [14], my results in this area give a new proof that the language $\mathbf{O}_2$ (and hence the rationally equivalent $\mathbf{MIX}_3 = \{w \in \{a,b,c\}^* : |w|_a = |w|_b = |w|_c\}$) associated to the group $\mathbb{Z}^2$ is generated by a 2-multiple context free grammar, a generalization of CFG. After recasting the statement in topological terms (where the words can be treated as knots in 2D), I give a homological proof, where the existence of a simplification (by which the language may inductively be proved to be MCF) follows by a degree argument for a map on a certain configuration space (see §2.1).

Future plans and opportunities for student research. This technique opens up the possibility of proving the analogous result in higher dimension using entirely topological tools, generalizing the result to smooth embedding of S^1 in $\mathbb{R}^n$ (knots) and consequently shedding light on the word problem in $\mathbb{Z}^n$. In recent years, a purely combinatorial proof was given by Salvati that answers the question affirmatively in the case of $\mathbf{O}_n$. A topological argument using Borsuk-Ulam theorem seems to be the obvious analogue and will possibly prove stronger properties about inscribing regular $2n$-sided polygons in knots in n-dimension. This research program would be of interest to students interested in Knot Theory or Computational Linguistics.

CURRENT AND PAST RESEARCH WITH UNDERGRADUATE STUDENTS AND OTHER OPPORTUNITIES.
As my research interests remain fairly broad in nature and include various unexplored but potentially tractable topics, I believe that there are numerous opportunities for undergraduate students to meaningfully participate and contribute to my program. Here are a couple of other areas of interest in which I have advised undergraduate students in the past, and would welcome further involvement from them in the future.

- **Tiling Invariants for Two-dimensional Families of regions** - The main problem in this area can be formulated as: *given a two-dimensional region, whether a set of tiles can be used to cover the region with non-overlapping placements* and if so, *how many different non-isomorphic ways there are*. Currently, I am supervising an undergraduate research project in this area that is related to *invariants for tiling annular regions with ribbon tiles*. The project uses Combinatorial Group Theory and Number Theory to produce mathematical invariants that persist through different configurations of tilings. An interesting question that remains open is, given $n \in \mathbb{N}$, whether or not it is possible to find a tile that admits precisely n distinct non-isomorphic tilings of the Euclidean plane.

- **Logic and Set Theory** - I have previously advised a Bachelor's thesis on *Cardinality and Continuum Hypothesis*. Further projects in this area may involve understanding Gödel's incompleteness theorem.

- **Differential Geometry and Hilbert's theorem** - This project is mostly expository in nature that explains the concept of curvature for abstract surfaces, explores the geometry of Hyperbolic space, and explains why it cannot be isometrically immersed in the Euclidean space.

- **Lake Surface Temperature Modeling and other Applied Math topics** - Another Bachelor's thesis I have advised in the past has been about creating a *Predictive model of Lake Surface Water Temperature* using differential equations and nonlinear regression techniques. I have also previously supervised an Independent Study on *Asset revenue modeling* using differential equations and machine learning.

- **Understanding the Math behind AI and an Illustrated Story Book generator** - This project is jointly with the CS department where the student is interested in creating a tool that will create a

short story given a text prompt, and will use the story to generate AI images. The math involved (attention theory, generative adversarial networks etc.) uses Linear Algebra, Multivariable Calculus, and Probability Theory, and the subject area is being actively developed. In fact, our primary resources were all published within the last five years!

- **Streamlining Numerical Analytic tools for practical applications** - Over the summer of 2022, I jointly supervised a research project through the *Applied Methods and Research Experience* program at the College of Wooster, funded by Goodyear Tire and Rubber Company - Innovation Technology division, where students were tasked with creating a comprehensive analysis application for their non-pneumatic tires using Python, converting multi-program routines involving complex data structures and cutting-edge numerical methods, into one standardized workflow.

- **Applied Data Science** - Over the summer of 2021, I jointly supervised another client-funded research project with a group of students; where students were tasked with understanding trends in customer behavior at a regional grocery store chain, analyzing halo effects using Data Science techniques, and coming up with creative targeted programs to increase sales using customer segmentation techniques (RFM analysis).

- A list of REU topics I supervised as a graduate student is available in my CV.

1.   NONABELIAN GROUP ACTIONS ON A CIRCLE AND EXTREMAL ROTATION NUMBER

The main goal of this research project is to study nonlinear actions of finitely generated groups on manifolds, in particular a circle, using the proper analog for character varieties. Since the real-valued rotation number $\widetilde{\mathrm{rot}} : \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathbb{R}$ defined as

$$\widetilde{\mathrm{rot}}(g) = \lim_{n \to \infty} \frac{g^n(0)}{n}$$

is a semi-conjugacy class invariant function (see e.g. Ghys [8] or Bucher-Frigerio-Hartnick [2]), it is reasonable to ask how $\widetilde{\mathrm{rot}}$ is constrained on the image of a finite subset of Γ under a homomorphism to $\widetilde{\mathrm{Homeo}}_+(S^1)$. Given $F = \langle a, b \rangle$, any word $w \in F$ and some representation $\rho : F \to \widetilde{\mathrm{Homeo}}_+(S^1)$, our main motivating question is as follows:

Research Program 1.1. For fixed values of $\widetilde{\mathrm{rot}}(\rho(a)) =: r$ and $\widetilde{\mathrm{rot}}(\rho(b)) =: s$, what is the set $X(w; r, s)$ of possible rotation number of w?

It is easy to prove that $X(w; r, s)$ is a compact interval whose extrema is achieved for some ρ and if we define $R(w; r, s) = \max\{X(w; r, s)\}$ then $\min\{X(w; r, s)\} = -R(w; -r, -s)$. So all the information about the program can be recovered by studying the function R, which we undertake in this project.

1.1.   POSITIVE WORDS AND CALEGARI-WALKER ZIGGURATS.

Theorem 1.2 (Calegari-Walker [3, Theorem 3.4, 3.7]). *Suppose w is positive i.e. w is in the semigroup generated by a and b (and not a power of a or b), and suppose r and s are rational. Then*

1. *$R(w; r, s)$ is rational with denominator no bigger than the smaller of the denominators of r and s; and*

2. *there is some $\epsilon(r, s) > 0$ so that $R(w; \cdot, \cdot)$ is constant on $[r, r + \epsilon) \times [s, s \times \epsilon)$.*

Furthermore, when r and s are rational and w is positive, Calegari-Walker give an explicit combinatorial algorithm to compute $R(w; r, s)$; it is the existence and properties of this algorithm that proves Theorem 1.2. Computer implementation of this algorithm allows one to draw pictures of the graph of R (restricted to $[0, 1) \times [0, 1)$) for certain short words w, producing a stairstep structure dubbed a *Ziggurat*; see Figure 1a.

In the special case of the word $w = ab$, a complete analysis can be made, and an explicit formula obtained for $R(ab; \cdot, \cdot)$ (this case arose earlier in the context of the classification of taut foliations of Seifert fibered spaces, where the formula was conjectured by Jankins-Neumann [10] and proved by Naimi [13]). But in *no other case* is any explicit formula known or even conjectured, and even the computation of $R(w; r, s)$ takes time exponential in the denominators of r and s.

1.1.1. Results. The *fringe* associated to w and a rational number $0 \leq p/q < 1$ is the interval $[s, 1)$ for which there is a homomorphism from F to $\widetilde{\mathrm{Homeo}}_+(S^1)$ with $r = p/q$ and

$$\widetilde{\mathrm{rot}}(\rho(w)) = \lim_{t \uparrow 1} R(w; p/q, t) = h_a(w)p/q + h_b(w).$$

Essentially, this is the length of the step starting inwards from the boundary at $(p/q, 1)$ before we drop down. The *fringe length* (see figure 1b), denoted $\mathrm{fr}_w(p/q)$, is equal to $1 - s$. My main result gives an explicit formula for fr_w for any positive word w and establishes a (partial) integral projective self-similarity for fringes, thus giving a theoretical basis for the experimental observations of Calegari-Walker and Gordenko [9].
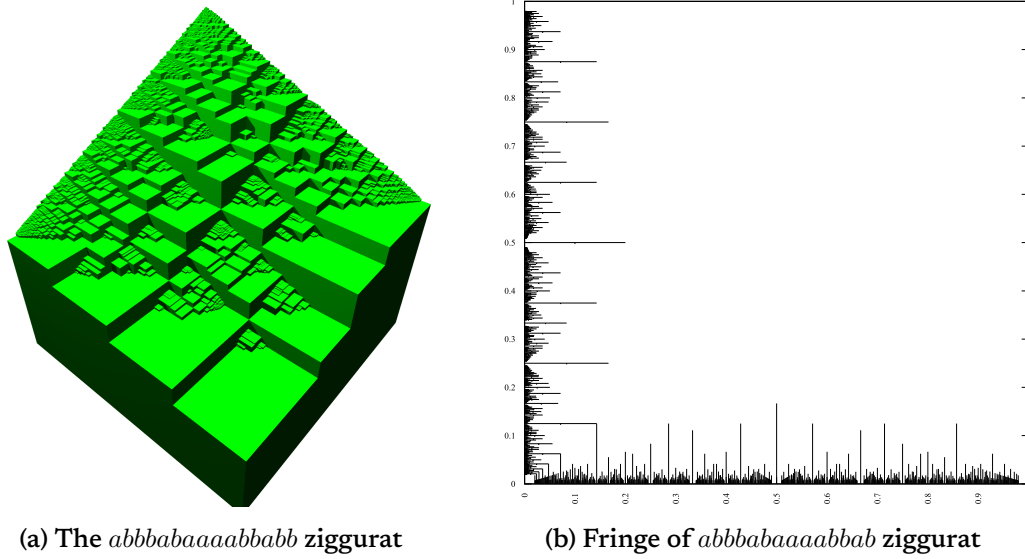
(a) The *abbbabaaaabbabb* ziggurat   (b) Fringe of *abbbabaaaabbab* ziggurat

Figure 1

Fringe Formula 1.3 (Chowdhury, [6]). *If w is positive, and p/q is a reduced fraction, then*

$$\mathbf{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q}$$

where $\sigma_w(g)$ depends only on the word w and $g := \mathbf{gcd}(q, h_a(w))$; and $g \cdot \sigma(g)$ is an integer.

The proof uses the stairstep algorithm by Calegari-Walker to transform the dynamical question to a linear programming problem, and using a theorem by Kaplan [11] and some combinatorial number theory we get the result.

As $t \to 1$, the dynamics of F on S^1 is approximated better and better by a linear model. For t close to 1, the non-linearity can be characterized by a perturbative model; fringes are the maximal regions where this perturbative model is valid. My theorem says that the size of this region of stability follows a power law. This is a new example of (topological) nonlinear phase locking in 1-dimensional dynamics giving rise to a power law, of which the most famous example is the phenomenon of Arnol'd Tongues [7].

The function $\sigma_w(g)$ depends on w and on q in a complicated but easily calculable way, and there are some special cases which are very easy to understand. In particular, I prove the following inequality:

σ-**inequality 1.4** (Chowdhury, [6]). *Suppose $w = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \ldots a^{\alpha_n} b^{\beta_n}$. Then the function $\sigma_w(g)$ satisfies the inequality*

$$\frac{h_b(w)}{h_a(w)} \leq \sigma_w(g) \leq \max \beta_i$$

Moreover, $h_b(w)/h_a(w) = \sigma_w(g)$ when h_a divides q, and $\sigma_w(g) = \max \beta_i$ when q and $h_a(w)$ are coprime.

Projective self-similiarity. The Fringe Formula explains the fact that $\mathrm{fr}_w(p/q)$ is independent of p (for $\gcd(p, q) = 1$) and implies a periodicity of fr_w on infinitely many scales. In particualr, I also show that the unit interval can be decomposed into some finite number of intervals Δ_i such that there exist a further decomposition of each Δ_i into a disjoint union of subintervals $I_{i,j}$ such that the graph of $\mathrm{fr}(x)$ on each of $I_{i,j}$ is similar to that on some $\Delta_{k(i,j)}$ under projective linear transformations.

1.1.2. Ongoing and Future Work. Although my formula gives an essentially complete characterization for the ziggurats near the boundaries, much is still unknown regarding the interior. For example, the following result that claims that nonlinearity of the extremal action corresponds to rigidity of the rotation number, remains unproven in general:

Conjecture 1.5 (Calegari-Walker). *Suppose w is a positive word of the form $w = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \ldots a^{\alpha_n} b^{\beta_n}$. If $R(w; r, s) = c/q$ where c/q is reduced, then*

$$|c/q - h_a(w)r - h_b(w)s| \leq n/q$$

It seems plausible that an approach using the stairstep algorithm and linear programming will lead to a possible resolution, and that is one of my future goals to pursue. Indeed using this technique, I have improved previously known bounds in some specific cases:

Proposition 1.6 (Chowdhury). *For any positive word w of the form $w = a^{\alpha_1} b^{\beta_1} \ldots a^{\alpha_n} b^{\beta_n}$, if $R(w; p/q, s) = c/q$ where p and c are coprime to q, we have the inequality*

$$|c/q - h_a(w)p/q - h_b(w)s| \leq n/q$$

Proposition 1.7 (Chowdhury). *For any positive word w of the form $w = a^{\alpha_1} b^{\beta_1} \ldots a^{\alpha_n} b^{\beta_n}$, if $R(w; 1/2, s) = c/d$ where c is coprime to d, we have the inequality*

$$|c/d - h_a(w)/2 - h_b(w)s| \leq n/d$$

The general case seems to be quite within reach. I have also been able to give bounds on the size of the stability region in the interior in specific cases such as follows:

Proposition 1.8. *If $(3, q) = 1$, then*

$$R\left(abaab; t, 1 - \frac{1}{q}\right) \text{ is constant} \quad \forall t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q^2}\right).$$

If $(3, q) \neq 1$, then

$$R\left(abaab; t, 1 - \frac{3}{2q}\right) \text{ is constant} \quad \forall t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{2q^2}\right).$$

The techniques used can be easily generalized to the case of prime q.

A related idea is that of slippery points [3]. A pair of rational numbers (r, s) is called *slippery* for a positive word w if there is a strict inequality $R(w; r', s') < R(w; r-, s-)$ for all $r' < r, s' < s$. It is important in this context because if conjecture 1.5 is true, then for any slippery point (r, s), we have $R(w; r-, s-) = h_a r + h_b s \in \mathbb{Q}$. In [3], the authors conjectured that $(1/2, 1/2)$ was expected to slippery for 'most' words. Continuing exploration in this direction, I was able to show that

Theorem 1.9 (Chowdhury). *Consider a positive word $w = b^{\beta_n} a^{\alpha_n} \cdots b^{\beta_1} a^{\alpha_1}$. Consider the image of the word under the quotient of F_2 by the infinite dihedral group $\langle a, b \mid a^2 = b^2 = 1 \rangle$. If the image is of the form $(ab)^n$ with $n \geq 1$, then $(1/2, 1/2)$ is not slippery for w.*
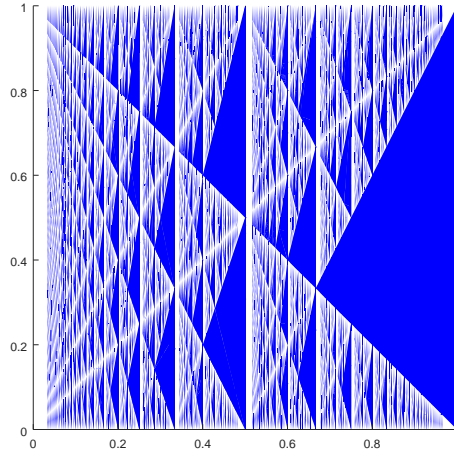
Figure 2: Recursive structure of the dense set of points where interval game can be won (§1.2.1)

1.2. GENERALIZATION TO ARBITRARY WORDS IN A FREE GROUP AND INTERVAL GAME.
Moving to the case of arbitrary words in free group, the first part of theorem 1.2 does hold for semi-positive words but in general, we have the conjecture

Conjecture 1.10 ([3]). *Let $w \in F$ be arbitrary and let $r, s \in \mathbb{Q}$. Then $R(w; r, s) \in \mathbb{Q}$.*

Since the monotonicity and stability results of ziggurats are not satisfied anymore, we use a different dynamical problem to analyze this case, called the *interval game*. Existence of a *winning* interval in this game ensures that conjecture 1.10 is true.

Goal 1.11. Given a finite collection of homeomorphisms $\varphi_1, \varphi_2, \ldots, \varphi_m$, called enemies, and a homeomophism ψ with irrational rotation number θ, we want find an interval I such that

- $\exists n \in \mathbb{N}$ such that $\psi^n(I^+) \in I$ and

- $\psi^i(I)$ is dijoint from $\varphi_j(I)$ for all $0 \le i \le n, q \le j \le m$.

1.2.1. Results. Calegari and Walker show that a winning interval exists when φ_i's are generic C^1 diffeomorphisms and for all cases of a single homeomorphism φ except when φ is a rigid rotation. The case when both ψ and φ are rigid rotations by θ and α, the set of points (θ, α) for which the game can be won, forms an open, dense subset of the unit square. In this context, the authors' description of the dense set and the proof had some minor mistakes , that I have fixed and the new dense set is pictured in figure 2.

I am currently working on extending the possible cases of winning scenarios to more than one homeomophism φ. In particular, the goal is to allow a wider variety, and eventually generic homeomorphisms that will have winning intervals. Currently I have shown that

Proposition 1.12. *Consider the interval game with two or more enemies φ_i and suppose rotation number of ψ is irrational. Let μ be an invariant probability measure for ψ. Suppose the following conditions are met:*

1. *there is a point $r \in S^1$ such that φ_i's are either weakly locally contarcting or expanding to the right of r.*

    2. the enemies that are weakly locally expanding to the right are locally Lipschitz to the right of r.

Then a winning interval exits.

Here by weakly locally contracting (resp. expanding) we mean that the graph of φ_i lies strictly below (resp. above) the the slope 1 line passing through $(r, \varphi_i(r))$.

1.2.2. Future Work. The immediate goal in future is to get around the Lipshitz condition in proposition 1.12 and generalize it further. I also would like to analyze the particular case of the word $aabAB$ in order to gain some insights about its extremal rotation numbers. Several geometric group theoretic tools including quasimorphism and stable commutator length are some of the most useful tools in this regard.

2. MULTIPLE CONTEXT FREE LANGUAGES AND SALVATI'S THEOREM

Given a finitely presented group G with finite generating set $\Sigma = \Sigma^{-1}$, the *word problem* asks whether a word in $\Sigma^* = \{a_1 a_2 \ldots a_n \mid a_i \in \Sigma, n \in \mathbb{N}\}$ is equal to the identity of G. The subset of Σ^* that satisfy the word problem is called the group language $L(G)$. Different presentations of the group give rationally equivalent group languages. A broad research program started by Anisimov [1] asks the following:

Research Program 2.1. Relate algebraic properties of finitely presented and finitely generated groups with language-theoretic properties of group languages.

There are several theorems in this area relating Chomsky hierarchies [4] of $L(G)$ to different class of groups, e.g. G is finite iff $L(G)$ is regular [1] and an infinite group G has more than one end if $L(G)$ is context-free [12]. In fact Muller and Schupp prove the stronger theorem that

Theorem 2.2 ([12]). *Let G be a finitely generated group. Then the following are equivalent.*

1. *G is virtually free.*

2. *$L(G)$ is context-free.*

3. *There exists a constant K such that every closed path in the Cayley graph $\Gamma(G)$ can be K-triangulated.*

A multiple context free grammar [17] deals with tuples of strings, with rewriting rules of the form

$$f[A_1, A_2, \ldots, A_n] \leftarrow A_0$$

where f is a function with tuples of strings as argument that satisfy

1. Each component of the value of f is a concatenation of some constant string and some component of its argument.

2. Each component is not allowed to appear in the value of f more than once.

Salvati [16] recently showed that the group language

$$\mathbf{O}_2 = \{w \in \{a, b, A, B\}^* \mid |w|_a = |w|_A \wedge |w|_b = |w|_B\}$$

and hence the rationally equivalent language

$$\mathbf{MIX}_3 = \{w \in \{a, b, c\}^* \mid |w|_a = |w|_b = |w|_c\}$$

are $2-$multiple context free i.e. they are generated by a 2-MCFG, which shows that they fall into the class of mildly context sensitive, the class that arguably best captures natural languages. The proof however has the disadvantage of using geometric ideas specific to two dimension e.g. Jordan curve theorem and complex exponential function, and as such no clear way of generalizing to higher dimensional variants. I give a new topological proof of this theorem using Homology theory of CW complexes, ideas not specific to 2-dimension, that raise possibility of generalization to higher dimensions.

2.1. A TOPOLOGICAL PROOF.

Salvati proves that the following grammar generates $\mathbf{O}_2$.

$$G = (\Omega, \mathcal{A}, R, S)$$

where $\Omega = (\{S; \mathbf{Inv}\}, \rho)$ with $\rho(S) = 1$ and $\rho(\mathbf{Inv}) = 2$; $\mathcal{A} = \{a, A, b, B\}$ and R consists of the following rules:

1. $S(x_1 x_2) \leftarrow \mathbf{Inv}(x_1, x_2)$

2. $\mathbf{Inv}(t_1, t_2) \leftarrow \mathbf{Inv}(x_1, x_2)$ where $t_1 t_2 \in \mathbf{perm}(x_1 x_2 a A) \cup \mathbf{perm}(x_1 x_2 b B)$

3. $\mathbf{Inv}(t_1, t_2) \leftarrow \mathbf{Inv}(x_1, x_2), \mathbf{Inv}(y_1, y_2)$ where $t_1 t_2 \in \mathbf{perm}(x_1 x_2 y_1 y_2)$

4. $\mathbf{Inv}(\epsilon, \epsilon)$

Since a word in O_2 corresponds uniquely to a closed path in the integer lattice $\mathbb{Z}^2$ (a and b correspond to $\rightarrow$ and $\uparrow$, and A, B to $\leftarrow$ and $\downarrow$ respectively), we prove a purely topological result about smooth closed curves in the plane that as a corollary gives Salvati's theorem.

Consider a closed (oriented) loop K in $\mathbb{R}^2$ given by

$$\varphi : S^1 \to \mathbb{R}^2, \quad Image(\varphi) =: K$$

It makes no difference to set the domain of φ equal to $[0, 1]$ and assume $\varphi(0) = \varphi(1)$. Let $p = \varphi(0), q = \varphi(1/2)$ and let r and s be two arbitrary points on K. Together the four points $\{p, q, r, s\}$ break up K into 4 (possibly degenerate) arcs K_1, K_2, K_3 and K_4 such that the starting point of K_{i+1} is the same as the ending point of K_i. Denote by $\vec{v}_i$ the vector which is defined as

$$\vec{v}_i = \text{End point of } K_i - \text{ Starting point of } K_i$$

The vectors $\vec{v}_i$ satisfy

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 = 0$$

My main technical result is the following.

Proposition 2.3 (Chowdhury, [5])**.** *Assume that φ is differentiable at p and q and $\varphi'(0)$ is not antiparallel to $\varphi'(1/2)$. Then there exists a pair of r and s on K different from $\{p, q\}$ such that the set of 4 vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ can be partitioned into two sets of 2 vectors each of which sum up to zero.*

The proof of Proposition 2.3 proceeds in multiple steps as follows. First we construct a 2 dimensional cell complex $\mathcal{X}$ whose points parametrizes choices of $\{r, s\} \neq \{p, q\}$ on K along with the particular choice of partition of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ into two sets of 2 vectors each. Then we define a function $f : \mathcal{X} \to \mathbb{R}^2$ with the property that a zero for f gives a choice of $\{r, s\}$ satisfying the conclusion of proposition 2.3. Finally using degree of f near link of certain vertices of $\mathcal{X}$, we show that f must have a zero. We subsequently prove Salvati's theorem using an induction argument and the following observation:

Lemma 2.4 (Chowdhury, [5])**.** *When φ is a closed path the the integer lattice $\mathbb{Z}^2$, we can choose r and s to be points with integer coordinates.*

2.2. ONGOING AND FUTURE WORK.

Recently in a paper, Salvati [15] proves that the language O_n is n-MCFL. His proof is entirely combinatorial and in particular uses the octohedral Tucker's Lemma. The natural topological analogue of this theorem is the Borsuk-Ulam theorem and so it seems reasonable to pursue a proof technique that uses topological tools. In particular, by proving the result topologically, we hope to generalize the theorem assertion to the case of loops (i.e. embedding of S^1) in $\mathbb{R}^n$.

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