Research Statement

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My primary research area is low dimensional topological dynamics, especially the theory of nonabelian group actions on the circle. I have also made contributions to the theory of formal languages, with an aim to solve combinatorial group theory problems using topological methods. I am broadly interested in geometric group theory and applied nonlinear dynamical systems and chaos theory related topics as well.

Extremal actions of free groups on the circle. The first research program can be roughly summarized as: *Given only the information about some dynamical invariant of a number of periodic processes, what information can we predict about the asymptotic behavior of their* ***composition*** *in some specific order*. We formulate the periodic processes below in terms of free group action on a circle, and the composition as the action of a word in the group.

The Poincaré rotation number, denoted $\widetilde{\text{rot}}$, is a real-valued semi-conjugacy invariant function on $\widetilde{\text{Homeo}}_+(S^1)$, the universal central extension of $\text{Homeo}_+(S^1)$, the group of orientation-preserving homeomorphisms of the circle. If w is a word in the free group $F \coloneqq \langle a.b \rangle$, given real numbers $r = \widetilde{\text{rot}}(\rho(a)), s = \widetilde{\text{rot}}(\rho(b))$, we would like to understand the set $X(w; r, s) \coloneqq \left\{\widetilde{\text{rot}}(\rho(w))\right\}$ and in particular, $R(w; r, s) \coloneqq \sup_\rho X(w; r, s)$, taken over all $\rho : F \to \text{Homeo}^\sim_+(S^1)$.

If w is in the semigroup generated by a and b, plotting the graph of R against r and s gives a staircase-like structure, called a Ziggurat. When $r = p/q$ is rational, it turns out that $R(w; p/q, t) = h_a(w)p/q + h_b(w)$ on an entire interval $t \in [1 - \text{fr}_w(p/q), 1)$, where h_a and h_b count the number a's and b's in w, and $\text{fr}_w(p/q)$ is a rational number, a priori depending on p/q in a complicated way. I proved that $\text{fr}_w(p/q) = \frac{1}{\sigma_w(g)\cdot q}$, where $g = gcd(q, h_b(w))$ and $\sigma_w(g) \cdot g \in \mathbb{Z}$ (with an algorithm for calculating it in $\mathcal{O}(1)$ time). This shows fr_w satisfies a *power law* and the maximal intervals are analogues of Arnol'd tongues in this context. This formula also explains the periodic nature of the ziggurat near the slopes.

Future plans and opportunities for student research. As my ongoing and future work in this project, I have plan to prove further rigidity and stability properties of this graph. For example, I have proved certain stability and rigidity results in the interior of the Ziggurat in some specific cases. In immediate future, I wish to understand more about the 'position' of the stairs in the Ziggurat and prove further bounds on the value of R. The opportunity for student involvement in this program comes from the fact that using some dynamical and number theoretic arguments we can reformulate the problem as a linear programming problem, which makes the tools and techniques more accessible to student researchers.

Additionally, the analogous problem for the general case of semi-positive and arbitrary words is still unexplored. Because of the nature of the problem, it is hard to formulate the process using a discrete model. I have proved and conjectured similar rationality results in this area using ideas from one-dimensional dynamics and the theory of Diophantine approximations. I expect that further progress in this area will use tools from Analysis and Nonlinear Dynamical Systems.

A Topological Approach to proving results in Formal Language theory. As the title suggests, this research program can be roughly summarized as: *Given a group theoretical formulation of certain hierarchical properties in mathematical linguistics, can we use topological tools such as homology and degree to answer whether a given word is trivial or not.*

A finitely generated group $G = \langle \Sigma \mid R \rangle$ is called context-free (CF) if its word problem $\{w \in \Sigma^\star \mid w =_R 1\}$ has a presentation where a set of replacement rules produce all possible strings in it via recursion. Such hierarchical properties in mathematical linguistics often relate to combinatorial and algebraic properties of groups and their Cayley graph in significant ways. Building on works of Salvati and Nederhof, my results in this area give a new proof that the language $\mathbf{O}_2$ (and hence the rationally equivalent $\mathbf{MIX}_3 = \{w \in \{a, b, c\}^* : |w|_a = |w|_b = |w|_c\}$) associated to the group $\mathbb{Z}^2$ is generated by a 2-multiple context free grammar, a generalization of CFG. After recasting the statement in topological terms (where the words can be treated as knots in 2D), I give a homological proof, where the existence of a simplification (by which the language may inductively be proved to be MCF) follows by a degree argument for a map on a certain configuration space.

Future plans and opportunities for student research. This technique opens up the possibility of proving the analogous result in higher dimension using entirely topological tools, generalizing the result to smooth embedding of S^1 in $\mathbb{R}^n$ (knots) and consequently shedding light on the word problem in $\mathbb{Z}^n$. In recent years, a purely combinatorial proof was given by Salvati that answers the question affirmatively in the case of $\mathbf{O}_n$. A topological argument using the Borsuk-Ulam theorem seems to be the obvious analog and will possibly prove stronger properties about inscribing regular $2n$-sided polygons in knots in n-dimension. This research program would be of interest to students interested in Knot Theory or Computational Linguistics.

Current and Past research with undergraduate students and other opportunities. As my research interests remain fairly broad in nature and include various unexplored but potentially tractable topics, I believe that there are numerous opportunities for undergraduate students to meaningfully participate and contribute to my program. Here are a couple of other areas of interest in which I have advised undergraduate students in the past, and would welcome further involvement from them in the future.

- **Tiling Invariants for Two-dimensional Families of regions** - The main problem in this area can be formulated as: *given a two-dimensional region, whether a set of tiles can be used to cover the region with non-overlapping placements* and if so, *how many different non-isomorphic ways there are*. Currently, I am supervising an undergraduate research project in this area that is related to *invariants for tiling annular regions with ribbon tiles*. The project uses Combinatorial Group Theory and Number Theory to produce mathematical invariants that persist through different configurations of tilings. An interesting question that remains open is, given $n \in \mathbb{N}$, whether or not it is possible to find a tile that admits precisely n distinct non-isomorphic tilings of the Euclidean plane.
- **Logic and Set Theory** - I have previously advised a Bachelor's thesis on *Cardinality and*

Continuum Hypothesis. Further projects in this area may involve understanding Gödel's incompleteness theorem.

- **Differential Geometry and Hilbert's theorem** - This project is mostly expository in nature that explains the concept of curvature for abstract surfaces, explores the geometry of Hyperbolic space, and explains why it cannot be isometrically immersed in the Euclidean space.
- **Lake Surface Temperature Modeling and other Applied Math topics** - Another Bachelor's thesis I have advised in the past had been about creating a *Predictive model of Lake Surface Water Temperature* using differential equations and nonlinear regression techniques. I have also previously supervised an Independent Study on *Asset revenue modeling* using differential equations and machine learning.
- **Understanding the Math behind AI and an Illustrated Story Book generator** - This project is jointly with the CS department where the student is interested in creating a tool that will create a short story given a text prompt, and will use the story to generate AI images. The math involved (attention theory, generative adversarial networks etc.) uses Linear Algebra, Multivariable Calculus, and Probability Theory, and the subject area is being actively developed. In fact, our primary resources were all published within the last five years!
- **Streamlining Numerical Analytic tools for practical applications** - Over the summer of 2022, I jointly supervised a research project through the *Applied Methods and Research Experience* program at the College of Wooster, funded by Goodyear Tire and Rubber Company - Innovation Technology division, where students were tasked with creating a comprehensive analysis application for their non-pneumatic tires using Python, converting multi-program routines involving complex data structures and cutting-edge numerical methods, into one standardized workflow.
- **Applied Data Science** - Over the summer of 2021, I jointly supervised another client-funded research project with a group of students; where students were tasked with understanding trends in customer behavior at a regional grocery store chain, analyzing halo effects using Data Science techniques, and coming up with creative targeted programs to increase sales using customer segmentation techniques (RFM analysis).
- A list of summer REU topics I supervised as a graduate student is available in my CV.