Notes for Rigidity Seminar

Gromov's Proof of Mostow Rigidity Theorem

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1 | INTRODUCTION

Given a homotopy equivalence $f : M \to N$, we have induced isomorphism of fundamental groups $\varphi : \Gamma \to \theta$ and the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow[\varphi-\text{equivariant}]{\widetilde{f}} & \mathbb{H}^n \\ \Big\downarrow p & & \Big\downarrow p \\ M = \Gamma\backslash\mathbb{H}^n & \xrightarrow{f} & \theta\backslash\mathbb{H}^n = N \end{array}$$

We want to talk about Gromov's proof of Mostow rigidity Theorem. The only similarity it has with Thurston's proof is that both use Boundary maps.

1.1 | Steps of the proof:

1. $\widetilde{f}$ is a pseudo isometry i.e.

$$\exists a, b \text{ such that } a^{-1}d(x, y) - b \leq d(\widetilde{f}(x), \widetilde{f}(y)) \leq ad(x, y)$$

2. g pseudo isometry $\rightsquigarrow$ continuous $g_+ : S^{n-1}_\infty \to S^{n-1}_\infty$ s.t. $\widetilde{f}_+$ is $\varphi-$equivariant.
3. If $v_0, v_1, \dots, v_n \in S^{n-1}_\infty$ span a geodesic $n-$simplex of maximal volume then so do $\widetilde{f}_+(v_0), \dots, \widetilde{f}_+(v_n)$.
4. $\widetilde{f}_+ = h_+$ for some isometry $h : \mathbb{H}^n \to \mathbb{H}^n$. $\rightsquigarrow$ $\varphi-$equiv. isometry h of $\mathbb{H}^n \rightsquigarrow \overline{h} : M \to N$.

Having already proved steps 1 and 2 in last lectures, we will start with proving step 3. However we will need to build some machinery before we proceed.

2 | GROMOV NORM

we introduce a homological invariant of a manifold known as Gromov's norm. Gromov's norm of hyperbolic manifolds will be seen to be proportional to the volume of the manifold. The first striking consequence of this result is that the volume of a hyperbolic manifold is a topological invariant.

Intuitively, Gromov's norm measures the efficiency with which multiples of a homology class can be represented by simplices. A complicated homology class needs many simplices.

Definition 2.1 (Gromov Norm)**.** Consider the homomorphism

$$i_* : H_2(S, \partial S; \mathbb{Z}) \to H_2(S, \partial S; \mathbb{R})$$

induced by inclusion map, and by abuse of notation, let $[S]$ denote the image of the fundamental class. Let $C = \sum_i r_i\sigma_i$ represent $[S]$, $r_i \in \mathbb{R}$. Denote

$$\|C\| = \sum_i |r_i| \text{ and the } \textbf{Simplicial Volume } \|S\| := \|[S]\| = \inf_C \|C\|$$

Note 2.2. Note that using $\|.\|$ one can define a seminorm on $H_k(S)$.

2.1 | Some properties of Gromov Norm

Proposition 2.3. *$f : X \to Y$ continuous and $\alpha \in H_k(X)$. Then*

$$\|f_*\alpha\| = \inf_{\substack{z\in C_k(Y)\\ [z]=f_*\alpha}} |z| \le \inf_{\substack{w\in C_k(X)\\ [w]=\alpha}} |f_*w| \le \inf_{\substack{w\in C_k(X)\\ [w]=\alpha}} |w| \le \|\alpha\|$$

since $f\circ\sigma_1$ and $f\circ\sigma_2$ may have same image.

Corollary 2.4. *M and N are homotopy equivalent $\implies \|M\| = \|N\|$.*

Proposition 2.5.

$$f : M \to N, \deg f = d \rightsquigarrow z = [M], f_*z = d[N] \implies |d|\|N\| \le \|f_*z\| \le \|z\| \rightsquigarrow d.\|N\| \le \|M\|$$

Corollary 2.6.

$$f : M \to M, \deg f = d > 1 \implies \|M\| = 0 \implies \|S^n\| = 0, \forall n \ge 1 \implies \|\alpha\| = 0, \forall\alpha \in H_1(X)$$

Note that $\|[S^0]\| = 2$. We will prove that $\|\text{Hyperbolic Manifold}\| > 0$.

Proposition 2.7.

$$f : M \to N \textit{ is a covering map}, \deg f = d \implies d.\|N\| = \|M\|$$
$$w = \sum c_i\sigma_i = [N] \rightsquigarrow \sum dc_i\sigma_i = [M] \implies \|M\| \le d\|N\|$$

Mostow Rigidity tells us that homotopy equivalent hyperbolic manifolds are isometric, so the geometric invariants (volume, diameter, injectivity radius) are somewhere encoded within the topology. More specifically, for any geometric invariant, there must be a topological invariant so that one can determine the geometric invariant by knowing the topological invariant. The following theorem tells us that the simplicial volume is the topological invariant corresponding to volume.

Theorem 2.8 (Gromov)**.** *Fix $n \ge 2$. Then there exists $v_n > 0$ such that for every closed hyperbolic manifold M*

$$\|M\| = \frac{Vol(M)}{v_n}$$

In fact, v_n is the maximum volume of a geodesic $n-$simplex in $\mathbb{H}^n$.

3 | STRAIGHTENING CHAINS

As we will now see, it is enough to take infimum over singular chains with geodesic simplices to calculate the Gromov norm.

Definition 3.1. Let M be a hyperbolic $m-$manifold, $\sigma : \Delta^n \to M$ be a singular $n-$simplex. Define *straightening* σ_g of σ as follows. First lift σ to a map from $\Delta^n \to \mathbb{H}^m$, denote it by $\widetilde{\sigma}$.

Let $v_0, \ldots, v_n$ denote the vertices of Δ^n. Consider the hyperboloid model

$$\mathbb{H}^m = \{x_{m+1} > 0 : x_1^2 + \ldots x_m^2 - x_{m+1}^2 = -1\}$$

For $v = \sum_i t_i v_i \in \Delta^n$ (barycentric coordinates), define

$$\widetilde{\sigma}_g(v) = \frac{\sum t_i \widetilde{\sigma}(v_i)}{-\| \sum t_i \widetilde{\sigma}(v_i) \|}$$

and $\sigma_g = p \circ \widetilde{\sigma}_g$. Intuitively, it is the projection of the convex hull of $\{v_0, \ldots, v_n\}$.

Define

$$str : C_*(M) \to C_*(M)$$

by setting $str(\sigma) = \sigma_g$ and extending by linearity. By composing a linear homotopy in $\mathbb{R}^{m+1}$ with radial projection to the hyperboloid, we see that there is a chain homotopy between str and Id. Thus $\|str(\alpha)\| \le \|\alpha\|$.

Lemma 3.2. *$v_k = \sup Vol(\sigma)$ over all geodesic simplices $\sigma : \Delta^k \to \mathbb{H}^n$ is finite for $k \neq 1$.*

Proof. It is enough to consider geodesic simplices with vertices on boundary (ideal simplex). Clearly, $v_2 = \pi$. Consider any ideal $k-$simplex σ in $\mathbb{H}^n$. Arrange σ so that one of its vertices is ∞ in upper half space model. So σ looks like a triangular chimney lying above a $(k-1)$ face σ_0 of σ. For $x \in \tau = pr(\sigma_0) \subseteq E^{n-1}$, let $h(x)$ denote the Euclidean height of σ_0 above x. Then volume of σ is

$$v(\sigma) = \int_\tau \int_h^\infty t^{-k} dt dEu^{k-1}$$

Then integrating we get

$$(k-1)v(\sigma) = \int_\tau h^{-(k-1)} dEu^{k-1}$$

The volume of σ_0 is obtained by a similar integral where dEu^{k-1} is replaced by Euclidean volume element of σ, which is $\ge dEu^{k-1}$. Thus

$$(k-1)v(\sigma) < v(\sigma_0) \le v_{k-1}$$

□

Haagerup-Munkholm proved that v_k is the volume of a regular ideal simplex. Milnor gave an asymptotic formula for v_n, assuming this result to be true. But we don't need them.

Corollary 3.3. *M closed hyperbolic $n-$manifold. Let $z = \sum c_i \sigma_i$ be a straight cochain representing $[M]$. Let $\Omega \in H^n(M)$ be the volume form. Then*

$$Vol(M) = \langle \Omega, [M] \rangle = \sum c_i Vol(\sigma_i) \le \sum c_i v_n \le v_n . \|z\| \Rightarrow \|M\| \ge \frac{Vol(M)}{v_n}$$

Thus we have proved one direction of Gromov's theorem.

3.1 Application to Hyperbolic surface

Theorem 3.4. *Let S be a compact orientable surface with $\chi(S) < 0$,* ***possibly with boundary.*** *Then*

$$\|S\| = -2\chi(S)$$

Proof. We know $\|S\| \ge -2\chi(S)$ from above theorem.

Lemma 3.5. *Let S be an orientable surface with p boundary components. If $p > 1$, then for any integer $m > 1$ with $(m, p-1) = 1$, there exists a $m-$fold cyclic cover S_m with p boundary components, each of which maps to the corresponding components of ∂S by a $m-$fold covering.*

Let $\chi(S) = 2-2g-p$. The surface S admits a triangulation with one vertex on each boundary component and no other vertex $\rightsquigarrow 4g+3p-4$ triangles $\rightsquigarrow$ a $m-$fold cover with $\chi(S_m) = 2m-2gm-mp$ and $p+m(4g+2p-4)$ triangles $\rightsquigarrow \|S\| = \frac{1}{m}\|S_m\| \leq \lim\limits_{m\to\infty} \frac{p+m(4g+2p-4)}{m} = -2\chi(S)$

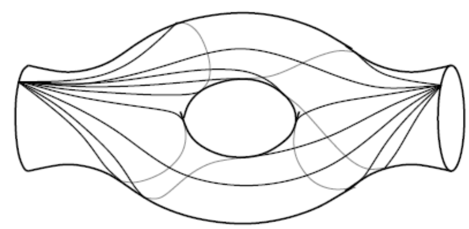

Figure 1: Example in case of $g = 1, p = 2$

□

4 REVERSE INEQUALITY IN GROMOV'S THEOREM

To prove the opposite inequality we need an explicit construction of a cycle representing $[M]$ and of total variation close to $Vol(M)/v_n$. The proof is easier to understand when using a modified homology which is a smoothing of singular homology. So we give an equivalent definition of Gromov norm, which is technically easier to work with.

4.1 Smooth Homology

Instead of continuous maps from the standard $k-$simplex, the chains of this modified homology are measures compactly supported on the space $C^1(\Delta^k, M) =: C^1_k$. More precisely, a $k-$chain μ is a signed Borel measure on $C^1(\Delta^k, M)$ with bounded total variation and $supp(\mu)$ is compact. Recall that any measure space (X, μ) admits a canonical splitting $(X_+, \mu_+), (X_-, \mu_-)$ such that $X_+ \cap X_- = \emptyset$ and $\mu = \mu_+ - \mu_-$ where μ_+ and μ_- are nonnegative measures. Then the total variation of μ is given by

$$\|\mu\|_v = \int_{X_+} d\mu_+ + \int_{X_-} d\mu_-$$

Let $\mathcal{C}^k(M)$ denote corresponding set of $k-$chains. the natural face inclusion $\eta_i : \Delta^{k-1} \to \Delta^k$ induce boundary maps in the following way. The map

$$\eta_i^* : C^1_k \to C^1_{k-1}$$

pushes forward to

$$\xi_i : \mathcal{C}^k(M) \to \mathcal{C}^{k-1}(M) \quad \mu \mapsto \xi_i(\mu) = (\eta_i^*)_*\mu$$

The boundary map is defined by $d_k = \sum(-1)^i \xi_i$.

We claim that the natural inclusion $i : C_*(M) \to \mathcal{C}_*(M)$ which sends σ to δ_σ is a chain map. Observe that on one hand

$$i(\partial\sigma) = \sum_{j=0}^{k}(-1)^j \delta_{\sigma_j}$$

where σ_j is the $j-$th side of σ. On the other hand,

$$\xi_j(i(\sigma))(A) = \delta_{\sigma_j}(A)$$

for all Borel subsets A of C^1_{k-1}. So $d(i(\sigma)) = \sum_{j=0}^{k}(-1)^j \delta_{\sigma_j}$.

Thus the inclusion map descends to a map between homology groups and in fact,

$$i_* : (H_n(C_*(M)), \|.\|) \to (\mathcal{H}_n(\mathcal{C}_*(M)), \|.\|_v)$$

is an isometric isomorphism. Note that integration of a $k-$form over an element of $\mathcal{C}_k(M)$ can be similarly defined via a deRham pairing.

4.2 Smearing Cocycle

Now we come to the explicit construction of the cycle mentioned above. We have a map of principal K bundles, where K is a maximal compact subgroup of $Isom^+(\mathbb{H}^n)$

$$\begin{array}{ccc} K & = & K \\ \downarrow & & \downarrow \\ Isom^+(\mathbb{H}^n) & \longrightarrow & \Gamma\backslash Isom^+(\mathbb{H}^n) = D(M) \\ \downarrow & & \downarrow \\ \mathbb{H}^n & \xrightarrow{\;p\;} & \Gamma\backslash\mathbb{H}^n = M \end{array}$$

and the horizontal maps are principal Γ bundles. As a topological space $Isom^+(\mathbb{H}^n) = K \times \mathbb{H}^n$ and the Haar measure h_0 on $Isom^+(\mathbb{H}^n)$ is the product of the one on K and the volume form $\Omega_{\mathbb{H}^n}$. Since h_0 is left invariant and $Isom^+(\mathbb{H}^n) \to D(M)$ is a locally trivial Γ-bundle, there is a unique measure h_M on $D(M)$ such that $Isom^+(\mathbb{H}^n) \to D(M)$ is locally measure preserving. Since, locally, h_M is the product of the Haar measure on K and the volume form Ω_M, one has

$$h_M(D(M)) = V(M)$$

One now defines a function

$$smear : C^1(\Delta^k, \mathbb{H}^n) \to \mathcal{C}_k(M)$$

as follows. Given $\sigma : \Delta^k \to \mathbb{H}^n$, there is a continuous map

$$\Psi : D(M) \to C^1_k(M)$$

given by

$$\Psi(\Gamma g) = p \circ g \circ \sigma, \quad g \in Isom^+(\mathbb{H}^n)$$

Definition 4.1. We define

$$smear(\sigma) = \Psi_*(h_M) \in \mathcal{C}_k(M)$$

The main property of $smear(\sigma)$ that we are interested in is

Lemma 4.2.

$$\|smear(\sigma)\|_v = vol(M)$$

if $\sigma \in C^1_n(\mathbb{H}^n)$.