Hilbert's 3^{rd} Problem and the Dehn Invariant

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Scissors Congruence

Two polygons P and Q are called Scissors Congruent in the plane if there exist finite sets of polygons $\{P_1, P_2, \ldots, P_m\}$ and $\{Q_1, Q_2, \ldots, Q_m\}$ such that the polygons in each respective set intersect with each other only on the boundaries, $\bigcup_{i=1}^{m} P_i = P$ and $\bigcup_{i=1}^{m} Q_i = Q$ and P_i is congruent to Q_i for each $i \in \{1, 2, \ldots, m\}$.

Theorem (Wallace-Bolyai-Gerwien Theorem)

Two polygons are Scissors Congruent if and only if they have the same area.

Proof.

LINK □

Hilbert's 3^{rd} Problem

Hilbert (1900)

given any two tetrahedra T_1 and T_2 with equal base area and equal height (and therefore equal volume), is it always possible to find a finite number of tetrahedra, so that when these tetrahedra are glued in some way to T_1 and also glued to T_2, the resulting polyhedra are scissors congruent?

Question (Reformulation)

Is it true that any two polyhedra of the same volume are scissors congruent?

Answer.

No. (Max Dehn, 1900)

Dehn Invariant

Definition

Consider the group $\mathbb{R}/\pi\mathbb{Q}$ with operation $+$ and identity 0. We want to focus on $\mathcal{V} = \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$. The Dehn invariant of a polyhedron P is defined as

$$D(P) = \sum \text{length}(e) \otimes [\theta(e)] \in \mathcal{V}$$

where $\theta(e)$ is the interior dihedral angle at the edge e and the sum is over all edges e of P.

Theorem

If P and Q are scissors congruent, then $\text{vol}(P) = \text{vol}(Q)$ and $D(P) = D(Q)$.

Answering Hilbert's question

Claim

A cube C and a tetrahedron T of unit volume are not Scissors Congruent.

Proof.

For a tetrahedron of volume 1, the length of each edge is $72^{1/3}$, and the measure of each angle is $\arccos(1/3)$. Thus

$$D(T) = \sum_{i=1}^{6} 72^{1/3} \otimes [\arccos(1/3)].$$

But $\arccos(1/3) \neq \mathbb{Q}$. So $D(T) \neq 0$. □

- Are volume and Dehn invariant sufficient to classify polytopes up to scissors congruence?
- What about other dimensions?
- What about other geometries, $\mathbb{H}^3$, $\mathbb{S}^3$ etc.?

Definition

$\mathcal{P}$ = the set of formal sums of all polyhedra with following group structure,

- $nP + mP = (n+m)P$
- $P = P_1 + P_2$ if P_1 and P_2 intersect only on edges or faces, and $P = P_1 \cup P_2$.
- $P = Q$ if P is congruent to Q.

Observe that, having $[P] = [Q]$ in $\mathcal{P}$ means that there exists a polyhedron A such that $P \cup A$ is scissors congruent to $Q \cup A$ and P and A (resp. Q and A) only intersect on their faces. Two such polyhedra are said to be *stably scissors congruent* which doesn't immediately imply that they are scissors congruent.

Theorem (Zylev)

For two polyhedra P and Q in $\mathbb{E}^3$, P is Scissors Congruent to Q if and only if P is stably Scissors Congruent to Q.

Prisms

Lemma

Two prisms P and Q are Scissors Congruent if and only if they have the same volume.

Theorem

All prisms have zero Dehn Invariant.

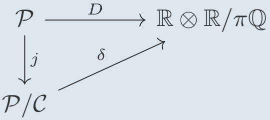
Proof.

Dihedral angles of orthogonal prisms are $\pi/2$. □

Let $\mathcal{P}/\mathcal{C}$ = the group of formal sums of all polyhedra modulo formal sums of prisms. This means that if you have a polyhedron P and a formal sum of prisms Q, such that P and Q do not intersect except on edges or faces, then P is equivalent to $P \cup Q$ in $\mathcal{P}/\mathcal{C}$.

Proposition

There exists a function $\delta : \mathcal{P}/\mathcal{C} \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$ such that $\delta \circ j(P) = D(P)$. That is, the following diagram commutes.



Theorem (Sydler)

If two polyhedra P and Q have the same volume and the same Dehn Invariant, then P and Q are scissors congruent.

Proof.

Assume δ is injective for now. $\implies [P] = [Q] \in \mathcal{P}/\mathcal{C}$
$\implies \exists$ prisms R and S s.t. R only intersects P on faces, S only intersects Q on faces, and $[P \cup R] = [Q \cup S] \in \mathcal{P}$.
$\implies P \cup R$ and $Q \cup S$ have the same volume.
$\implies \mathrm{vol}(R) = \mathrm{vol}(S)$.
But $[R] = [S]$. So $[P] = [Q] \in \mathcal{P}$. □

Why is δ injective?

Theorem

Let ϕ *be a homomorphism* $\phi : \mathbb{R} \to \mathcal{P}/\mathcal{C}$ *such that*

1. $\phi(a+b) = \phi(a) + \phi(b)$
2. $\phi(na) = n\phi(a)$ *for* $n \in \mathbb{Z}$
3. $\phi(\pi) = 0$
4. $[T] = \sum_{i=1}^{6} \text{length}(e_i)\phi(\theta_i) \in \mathcal{P}/\mathcal{C}$

We follow the construction of Zakharevich for ϕ*. Suppose* $\exists\, h : (0,1) \to \mathcal{P}/\mathcal{C}$ *s.t.*

$$[T(a,b)] = h(a) + h(b) - h(a,b) \text{ and } ah(a) + bh(b) = 0 \text{ if } a+b=1$$

Then

$$\phi(\alpha) = \tan(\alpha) \cdot h(\sin^2(\alpha))$$

where $(n\pi)/2 = 0$*. We claim that such a function exists.*

Let $\mathcal{D}(X) = Ker(\Delta : \mathcal{P}(X) \to \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q})$. Then

1. Conjecture: $vol : \mathcal{D}(\mathbb{H}^3) \to \mathbb{R}$ and $\mathcal{D}(\mathbb{S}^3) \to \mathbb{R}$ are injective.
2. Theorem: Then have countable image. in fact they are $\mathbb{Q}$ vactor space of countable dimension.
3. Higher dimension.
4. Mixed Dimension.