GAUSSIAN CURVATURE AND THE GAUSS-BONNET THEOREM

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1. Preamble

The Gauss-Bonnet Theorem is one of the most beautiful and one of the deepest results in the differential geometry of surfaces. It concerns a surface S with boundary ∂S in Euclidean $3-$space, and expresses a relation between:

- the integral of the Gaussian curvature over the surface,
- the integral of the geodesic curvature of the boundary of the surface, and
- the topology of the surface, as expressed by its Euler characteristic.

Note. If you have some familiarity with the material, you are under no obligation to attempt the **(B)** exercises, but you should at least convince yourself that you know how to do them.

2. Basic Differential Geometry of Surfaces

Proposition 2.1. *Let (ϕ, U) be a local chart for a surface M and let $q \in U$ with $p = \phi(q) \in M$. Then*

$$T_pM = (D(\phi(q)))(\mathbb{R}^2)$$

This implies that $(\phi_u(q), \phi_v(q))$ forms a basis for T_pM, with (u, v) coordinates in U.

Definition 2.2. The restriction of the canonical inner product in $\mathbb{R}^3$ to T_pM is called the first Fundamental Form (at p), denoted I_p.

Consider an element of T_pM, a tangent vector at $p = \gamma(0) \in M$ to a parametrized curve $\gamma(t) = \phi(u(t), v(t)), t \in (\varepsilon, \varepsilon)$. Then,

$$\mathrm{I}_p(\gamma', \gamma') = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

where $E = \langle \phi_u, \phi_u \rangle$ etc.

Definition 2.3. A regular surface M is orientable if it is possible to cover it with an atlas $\mathcal{A}$, so that $\forall i, j \in I$ and $\forall p \in U_i \cap U_j$

$$\det(\phi_j \circ \phi_i^{-1})(\phi_i(p)) > 0$$

We will be using the Darboux-Carter frame of reference, constructed by orthonormal vectors $(T, n, B \coloneqq T \times n)$ where $T = \gamma'$ and n is a unit normal to M at $p \in \gamma$.

Definition 2.4. The Gauss map (on an oriented surface) is the map $n : M \to S^2 \subset \mathbb{R}^3$ defined as

$$n(p) = \frac{\phi_u \times \phi_v}{||\phi_u \times \phi_v||}(p)$$

Thus $n(p)$ is a unit vector orthogonal to T_pM.

Date: September 14, 2017.

Definition 2.5. The Weingarten Map is $L_p : T_pM \to T_pM$ defined as

$$L_p(v) = -Dn(p)(v) = -\frac{d}{dt}(n \circ \gamma)(0)$$

Definition 2.6. The second fundamental form of a regular oriented surface M at a point p is the inner product on T_pM defined as

$$\mathrm{II}_p(X,Y) = \langle L_p(X), Y\rangle$$

➪**Exercises 2.7.** (1) **(B)** Prove that L_p is self adjoint. [H: L_p is linear]

(2) Consequently show that the Weingarten map is self adjoint.
(3) Prove that

$$\mathrm{II}_p(\gamma', \gamma') = e\left(\frac{du}{dt}\right)^2 + 2f\left(\frac{du}{dt}\frac{dv}{dt}\right) + g\left(\frac{dv}{dt}\right)^2$$

where $e = -\langle n_u, \phi_u\rangle = -\langle n, \phi_{uu}\rangle$ etc.

Definition 2.8. The *Gaussian Curvature* is defined as

$$K(p) = \det(Dn(p)).$$

Thus it is the product of the two principal curvatures at p which in turn are the traces of the matrix $Dn(p)$.

➪**Exercises 2.9.** (1) **(B)** Show that for a round 2−sphere of radius r about the origin, $L_p = \frac{1}{r}Id$. Consequently, what is the Gaussian curvature of a sphere?

(2) Let S be a quadratic surface $z = ax^2 + by^2$, parametrize S by the map $X : \mathbb{R}^2 \to S$ given by

$$X(u,v) = (u, v, au^2 + bv^2)$$

(a) Evaluate X_u, X_v, n.
(b) Check that $Dn(p)(X_u) = n_u$ and evaluate it.
(c) Check that n_u and n_v are orthogonal to n.
(d) Show that at $p =$ origin, the map $Dn(p)$ is given by $\begin{pmatrix} -2a & 0 \\ 0 & -2b \end{pmatrix}$ w.r.t. the basis $X_u = (1,0,0)$ and $X_v = (0,1,0)$ of $T_{origin}S$.

(3) Consider the helicoid parametrised as

$$X(u,v) = (u\cos v, u \sin v, v)$$

Do similar steps as above and prove that

$$Dn(X_u) = n_u = \frac{X_v}{(1+u^2)^{3/2}} \text{ and } Dn(X_v) = n_v = \frac{X_u}{(1+u^2)^{1/2}}$$

Theorem 2.10 (Theorema Egregium)**.** *The Gaussian curvature $\kappa(p)$ can be formulated entirely using I_p and its first and second derivatives. As such, it is an intrinsic value of the surface itself at p, i.e. it does not depend on the embedding of the surface in $\mathbb{R}^3$ and depends only on t he metric tensor g at p.*

Proof of this result uses Christoffel symbols which we will not go into in this note.

3. Gauss-Bonnet Theorem

Definition 3.1. The geodesic curvature of a regular curve γ on a regular surface S, at a given point p is defined as

$$\kappa_g(p) := \langle \gamma'', B(p) \rangle$$

➪Exercises 3.2. (1) **(B)** Show that

$$\gamma'' = \kappa_g B + \kappa_n n$$

where $\kappa_n := \langle \gamma'', n \rangle$ is the normal curvature of the curve γ.

(2) Consider the circle of latitude θ around the sphere of radius 1. What's the geodesic curvature of this curve?

Theorem 3.3 (Gauss-Bonnet Formula)**.** *Let (M, n) be an oriented surface with $M \subset \mathbb{R}^3$ and let (ϕ, U) be a coordinate patch with $\phi : U \to \mathbb{R}^3, \phi(U) \subset M$.*

Let γ be a piecewise regular curve on M enclosing a region $R \subset M$.

Let $\{\gamma_i\}_{i=1}^n$ be the regular curves that form γ and denote by $\{\alpha_i\}_{i=1}^n$ the 'jump' angles at the junction points (exterior angles).

Then we have

$$\int_R K\,dA + \int_\gamma \kappa_g\,ds = 2\pi - \sum_{i=1}^n \alpha_i$$

➪Exercises 3.4. (1) **(B)** Check that Gauss-Bonnet formula holds for the polar cap enclosed by the circle of latitude θ as above.

(2) **(B)** Prove (without using Gauss-Bonnet formula) that area of geodesic triangle on the unit $2-$sphere S^2 with interior angles α, β, and γ is given by $T = \alpha + \beta + \gamma - \pi$. Check that this is consistent with above formula.

(3) Let $M \subset \mathbb{R}^3$ be an oriented compact regular surface, K its Gaussian curvature and χ its Euler characteristic. Then prove that

$$\int_M K\,dA = 2\pi\chi(M)$$

Here are the steps:

(a) Consider a triangulation of M and apply the G-B formula to each of the triangles.
(b) Summing over all triangles cancels out the κ_g terms.
(c) Conclude that

$$\int_M K\,dA = 2\pi V - \pi F$$

(d) Show that $2\pi V - \pi F = 2\pi\chi(M)$.

The most general statement of the Gauss-Bonnet theorem is given by

$$\int_M K\,dA + \int_{\partial M} \kappa_g\,ds = 2\pi\chi(M)$$

for a compact Riemmannian manifold M with boundary ∂M. If the boundary ∂M is piecewise smooth, then we interpret the integral $\int_{\partial M} k_g\,ds$ as the sum of the corresponding integrals along the smooth portions of the boundary, plus the sum of the angles by which the smooth portions turn at the corners of the boundary.

⇨**Exercises 3.5.**

(1) **(B)** Let $S \subset \mathbb{R}^3$ be a smooth closed surface (this is automatically orientable; why?) which is *not* homeomorphic to S^2. Show that there are points on S where the Gaussian curvature is positive, zero and negative.

(2) Let S be a torus of revolution in $\mathbb{R}^3$. Visualize the image of the Gauss map, and see directly (without using the Gauss-Bonnet Theorem) that

$$\int_S K\, dA = 0$$

(3) Let S be a smooth surface homeomorphic to S^2. Suppose $\Gamma \subset S$ is a simple closed geodesic, and let A and B be the two regions on S with Γ as boundary. Let $n : S \to S^2$ be the Gauss map. Prove that $n(A)$ and $n(B)$ have the same area on S^2.

(4) Let $S \subset \mathbb{R}^3$ be a surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 on S which start at a point p can not meet again at a point q in such a way that together they bound a region S' on S which is homeomorphic to a disk.

(5) Let $S \subset \mathbb{R}^3$ be a surface homeomorphic to a cylinder and with Gaussian curvature $K < 0$ (e.g. a hyperboloid of one sheet). Show that S has at most one simple closed geodesic.

(6) Let $S \subset \mathbb{R}^3$ be a smooth closed surface of positive curvature, and thus homeomorphic to S^2. Show that if Γ_1 and γ_2 are two simple closed geodesics on S, then they must intersect one another.

(7) If M is a closed orientable surface in $\mathbb{R}^3$ with $K > 0$ everywhere, then prove that M must be convex.

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