FUNDAMENTAL GROUPS AND COVERING SPACES

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1. Preamble

The following exercises are intended to introduce you to some of the basic ideas in algebraic topology, namely the fundamental group and covering spaces. Exercises marked with a **(B)** are basic and fundamental. If you are new to the subject, your time will probably be best spent digesting the **(B)** exercises. If you have some familiarity with the material, you are under no obligation to attempt the **(B)** exercises, but you should at least convince yourself that you know how to do them. If you are looking to focus on specific ideas and techniques, I've made some attempt to label exercises with the sorts of ideas involved.

2. Two extremely important theorems

If you get nothing else out of your quarter of algebraic topology, you should know and understand the following two theorems. The exercises on this sheet (mostly) exclusively rely on them, along with the ability to reason spatially and geometrically.

Theorem 2.1 (**Seifert-van Kampen**). *Let X be a topological space and assume that $X = U \cup V$ with $U, V \subset X$ open such that $U \cap V$ is path connected and let $x_0 \in U \cap V$. We consider the commutative diagram*

$$\begin{array}{ccccc} & & \pi_1(U, x_0) & & \\ & \overset{i_{1*}}{\nearrow} & & \overset{j_{1*}}{\searrow} & \\ \pi_1(U \cap V, x_0) & & & & \pi_1(X, x_0) \\ & \underset{i_{2*}}{\searrow} & & \underset{j_{2*}}{\nearrow} & \\ & & \pi_1(V, x_0) & & \end{array}$$

where all maps are induced by the inclusions. Then for any group H and any group homomorphisms $\phi_1 : \pi_1(U, x_0) \to H$ and $\phi_2 : \pi_1(V, x_0) \to H$ such that

$$\phi_1 i_{1*} = \phi_2 i_{2*},$$

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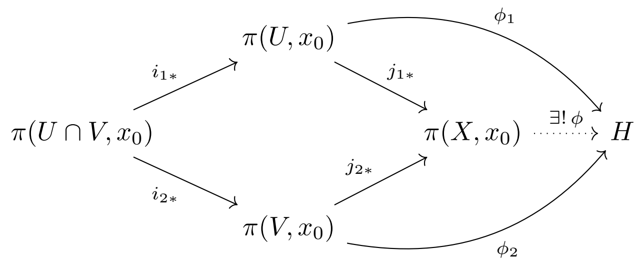
there exists a unique group homomorphism $\phi : \pi_1(X, x_0) \to H$ *such that* $\phi_1 = \phi j_{1*}$, $\phi_2 = \phi j_{2*}$.



Although it may not be clear from the statement, $\pi_1(X, x_0)$ is uniquely determined in terms of $\pi_1(U, x_0), \pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$ by the theorem. In section 4, we describe some further ideas needed to observe this and restate the theorem in a more familiar form.

Theorem 2.2 (**Galois correspondence**). *For any reasonably nice topological space X, there is a one-one correspondence*

$$\left\{ \begin{array}{c} \textit{Conjugacy classes of} \\ \textit{subgroups } H \leq \pi_1 X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{covering spaces } f : \tilde{X}_H \to X \end{array} \right\}.$$

The correspondence proceeds by associating a subgroup H with a cover $\tilde{X}_H$ for which $f_*(\pi_1 \tilde{X}_H)$ is conjugate to H in $\pi_1 X$.

If $H \lhd \pi_1 X$ is normal, then there is an action of $Q = \pi_1 X/H$ on $\tilde{X}_H$. This action is free and properly discontinuous, and the quotient $\tilde{X}_H/Q$ is homeomorphic to X.

3. CW complex and Cell decomposition

Most "nice" topological spaces can be given a skeletal structure. For example, one familiar way of constructing the torus $\mathbb{T}^2 = S^1 \times S^1$ is by identifying opposite sides of a square. Thus the 4 edges become two intersecting circles in the torus. Thus we can think of $\mathbb{T}^2$ obtained as gluing a disk onto this $1-$dimensional structure (the 1-skeleton) via some map on the boundary. A natural generalization goes as follows:

Step 1 : Start with a discrete set of points, called the $0-$skeleton.

Step 2 : Inductively, form the n skeleton X^n from X^{n-1} by attaching n-disks D^n's called n-cells e^n, via maps $\chi_\alpha : S^{n-1} = \partial D^n \to X^{n-1}$.

Step 3 : Take $X = X^n$ for finite n or $X = \cup_n X_n$ with the weak topology if continuing indefinitely.

Spaces constructed this way are called *CW complexes* or *cell complexes.*

4. Free products and Amalgamation of groups

We suppose we are given a set S, and a collection of groups $\{G_s\}_{s \in S}$. The idea of the free product of the G_s is to take the union of the generators and relations for the G_s, with no additional relations. We give a precise definition in the case S is finite. Obvious generalizations can be made in the infinite case using presentations of groups.

Definition 4.1. The free product $G * H$ of groups G and H is the set of elements of the form

$$g_1h_1g_2h_2...g_rh_r,$$

where $g_i \in G$ and $h_i \in H$, with g_1 and h_r possibly equal to e, the identity element of G and H.

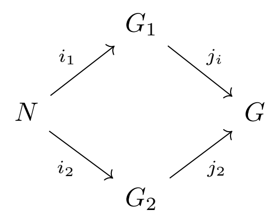
Free products of more than two groups are defined recursively, i.e.,

$$G_1 * G_2 * ... * G_n = \big(G_1 * G_2 * ... * G_{(n-1)}\big) * G_n.$$

The free group F_n is the free product of $\mathbb{Z}$ with itself n times.

➩**Exercise 4.2. (B)** Give a group presentation of $\mathbb{Z}_4 * \mathbb{Z}_5$.

Free products of groups are generalized by a notion of amalgamated products of groups joined together along specified subgroups. Let N, G_1, G_2 be groups and $i_1 : N \to G_1, i_2 : N \to G_2$ be group homomorphisms. For a triple (G, j_1, j_2) such that



we say that it satisfies the universal property if for any other triple (H, ϕ_1, ϕ_2) as above, there exists a unique group homomorphism $\phi : G \to H$ such that $\phi_1 = \phi j_1$ and $\phi_2 = \phi j_2$.

Definition 4.3. The *amalgamated free product* associated to (N, G_1, G_2, i_1, i_2) is a triple (G, j_1, j_2) satisfying this universal property. We denote G as $G_1 *_N G_2$ (keep in mind the maps involved!).

Amalgamated free product is unique and is constructed as follows: let $\langle\langle N\rangle\rangle$ be the normal subgroup of $G_1 * G_2$ generated by elements of the form $i_1(n)i_2(n)^{-1}$ for $n \in N$; then

$$G_1 *_N G_2 := (G_1 * G_2)/\langle\langle N\rangle\rangle.$$

Thus a reformulation of the Seifert-van Kampen theorem tells us that

$$\pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0)$$

where the maps involved are induced by inclusions.

➩**Exercise 4.4.** Think about the case $G_1 = G_2 = \{1\}$.

(1)**(B)** Prove that $\pi_1(S^n, p) = 1$ for all $n \geq 2$ and all $p \in S^n$.
(2)**(B)** Deduce that $\pi_1(\mathbb{RP}^n, p)$ is isomorphic to $\mathbb{Z}_2$ for all $n \geq 2$ and $p \in \mathbb{RP}^n$.
(3)**(B)** What is $\pi_1(\mathbb{RP}^1, p)$?

4.1. **General Strategy to find the Fundamental group of a CW complex.**

Step 1 : If X is homotopic to a simpler X', then use that instead.

Step 2 : The case $N = G_2 = \{1\}$ tells us an interesting fact, namely if we have a cell decomposition of a space X, then we can throw away the n-cells for $n \geq 3$ for calculating the fundamental group.[1]

Then what about $2-$cells? We can account for them using the case when $G_2 = 1$. We get the following proposition

Proposition 4.5. *Assume X is obtained from A by adjoining a two cell e with characteristic map $\chi_e : S^1 \to A$. Let $a = \chi_e(1)$, and let $\alpha \in \pi_1(A, a)$ denote the class of the path $t \mapsto \chi_e(\exp^{2\pi i t})$. Then*

$$\pi_1(X, a) \cong \pi_1(A, a)/\langle \alpha \rangle.$$

➪**Exercise 4.6.** Prove this Proposition.

Step 3 : Use above proposition to get rid of $2-$cells. Thus we find that the fundamental group we are interested in is isomorphic to the quotient of $\pi_1(X_1)$ by the smallest normal subgroup generated by paths induced by $\chi_{2-cells}$.

Step 4 : Once you are in X_1, van Kampen theorem should suffice.

Note that this theorem is a special case of the fact that spaces which are homotopy equivalent have isomorphic fundamental groups, and thus, helps in executing **Step 1**.

➪**Exercise 4.7.** You will need exercise 5.4.2 to do the following.

(1) **(B)** Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.
(2) **(B)** Calculate the fundamental groups of a Torus, a Klein Bottle and $\mathbb{RP}^2$.

We finish this section by citing another important idea in geometric group theory called the HNN-extension.It is easy to find out the U and V in Seifert-van Kampen theorem for example when you are cutting a surface X along a separating curve to get U and V. What happens when you cut it along a non-separating curve? What if $U \cap V$ is not connected?

Definition 4.8. Suppose $\phi, \psi : N \hookrightarrow A$ are both injective homomorphisms. If A has presentation $\langle S \mid R \rangle$ then the Higman-Neumann-Neumann (HNN) extension is

$$A*_N \cong \Big\langle S, t \mid R, \{t\psi(n)t^{-1} = \phi(n) \mid n \in N\} \Big\rangle$$

When ψ is just the identity homomorphism induced by inclusion, we may also denote the HNN-extension as $A*_\phi$.

Take a look at section 8 to find out how they arise as fundamental groups.

[1]Thus if X is a compact, path connected space then, for any cell decomposition of X, the inclusion of the $2-$skeleton X^2 into X, $i : X_2 \to X$, induces an isomorphism of groups (for any basepoint $x \in X_2$)

$$i_* : \pi_1(X^2, x) \xrightarrow{\sim} \pi_1(X, x).$$

5. Algebraic topology of graphs

The following exercise will explore some aspects of the algebraic topology of graphs. For our purposes, a *graph* is a topological space X consisting of some (possibly infinite) number of copies of the unit interval $I = [0, 1]$, with certain identifications of endpoints (via the quotient topology). For example, the circle S^1 is a graph consisting of one copy of I with endpoints identified. To study the algebraic topology of graphs, we will require the following preliminary results.

Basic, super-important theorem: $\pi_1(S^1) = \mathbb{Z}$.

Definition 5.1. A topological space X is said to be *contractible* if there is a (continuous) map $f : X \times [0, 1] \to X$ such that

(1) $f(\cdot, 0) = \mathrm{id}$
(2) $f(\cdot, 1)$ is constant.

One could equivalently say that X is contractible if it deformation retracts to a point (see Section 7), or if the identity map is homotopic to a point.

Theorem 5.2. *If X is contractible, then $\pi_1(x) = \{1\}$, the trivial group.*

⇨**Exercise 5.3. (B)** Prove this! (Definitions)

⇨**Exercises 5.4.**

(1) **(B)** Prove that a tree (in the graph-theoretic sense) is contractible. (Definitions)
(2) **(B)** Show that the fundamental group of the following graph X is the free group on two generators a, b (van Kampen):

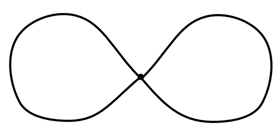


Figure 1. Figure-8

(3) **(B)** Is the following graph a covering space of X as in (2)? (Definitions)

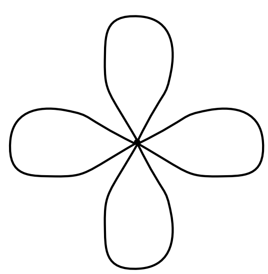


Figure 2. Topological Rose

(4) **(B)** Is the following graph Y a covering space of X as in (2)? (Definitions)

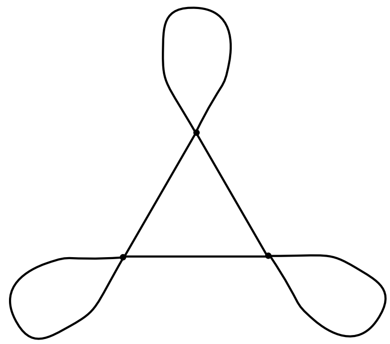


FIGURE 3.

(5) **(B)** If so, find a subgroup $H \leq F_2$ that corresponds to a covering $Y \to X$. (Galois correspondence)
(6) Is the subgroup of the previous question normal in F_2? If so, describe a homomorphism $f : F_2 \to G$ with kernel H. (Galois correspondence; normal subgroup case)
(7) Is every index-3 subgroup of F_2 normal?

6. Group Actions and Galois correspondence

The immediate goal of the theory of covering spaces developed here is to use them to compute $\pi_1(X)$.

6.1. **Group Actions.** A (left) action of a (topological) group G on a topological space X consists of a (continuous) map $f : G \times X \to X$ s.t.

(1) $ex = x \quad \forall x \in X$,
(2) $g_1(g_2x) = (g_1g_2)x \quad \forall g_1, g_2 \in G, x \in X$.

Thus given a group G and a space X , an action of G on X is a homomorphism from G to the group Homeo(X). We shall be interested in actions satisfying the following condition:

Each $x \in X$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint, i.e $g_1(U) \cap g_2(U) \neq \emptyset$ implies $g_1 = g_2$. These are called *covering space actions.*

If an action of a group G on X satisfies this condition, then $p : X \to X/G$ is a covering map.

Assuming the *path lifting* and *homotopy lifting properties* we arrive at the following theorem:

Theorem 6.1. *For a covering space action of a group G on a simply-connected space X the fundamental group $\pi_1(X/G)$ is isomorphic to G.*

6.2. **Galois correspondence.** An isomorphism between covering spaces $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ is a homeomorphism $f : E_1 \to E_2$ such that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow{\cong} & X \end{array}$$

The materials we are building in this section will eventually give us a tool to classify covering spaces of a given topological space X up to isomorphism, known as the *Galois correspondence*.

It follows from the path lifting and homotopy lifting properties of covering maps that:

Proposition 6.2. *Let $p : (E, e_0) \to (X, x_0)$ be a covering projection. Then $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is a monomorphism.*

Theorem 6.3 (**Lifting criterion**). *Let $(Y, y_0) \to (X, x_0)$ be a map of pointed spaces, Y being path connected and locally path connected, and $p : (E, e_0) \to (X, x_0)$ being a covering space. Then f has a lift*

$$\tilde{f} : (Y, y_0) \to (E, e_0)$$

iff

$$f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0).$$

Moreover, this lift is unique.

Now with the help of the unique lifting property for covering projections, we get the uniqueness half of the Galois correspondence:

Theorem 6.4. *Let X be a path connected and locally path connected space, x_0 its base point. Two path connected covering spaces $p_1 : (E_1, e_1) \to (X, x_0)$ and $p_2 : (E_2, e_2) \to (X, x_0)$ are isomorphic as pointed spaces via an isomorphism $f : (E_1, e_1) \to (E_2, e_2)$ iff*

$$p_{1*}\pi_1(E_1, e_1) = p_{2*}\pi_1(E_2, e_2).$$

E_1 and E_2 are isomorphic without regard to basepoints iff $p_{1*}\pi_1(E_1, e_1)$ and $p_{2*}\pi_1(E_2, e_2)$ are conjugates in $\pi_1(X, x_0)$.

For an arbitrary covering space $p : E \to X$ one can consider the isomorphisms from this covering space to itself, called *deck transformations* or *covering transformations*, and these form a group under composition called the *Deck group*, denoted by $G(E)$. Noting that the action of $G(E)$ on E is a covering space action *(prove it!)*, we can now conclude with the existence part of the Galois correspondence:

Theorem 6.5. *If a path-connected, locally path-connected space X has a simply connected covering space, then every subgroup of $\pi_1(X, x_0)$ is realized as $p_*\pi_1(E, e_0)$ for some covering projection $p : (E, e_0) \to (X, x_0)$.*

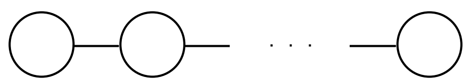
Definition 6.6. A covering space $p : E \to X$ is **normal** if for each point x in X and each pair of points e_1 and e_2 in the fiber of x, there is a Deck transformation taking e_1 to e_2.

Equivalently, a covering space is normal if its fundamental group is isomorphic to a normal subgroup of the fundamental group of the base space (prove this!). A good exercise at this point will be to try solving Exercise 4.4 (2) and (3), and (5) from section 8.

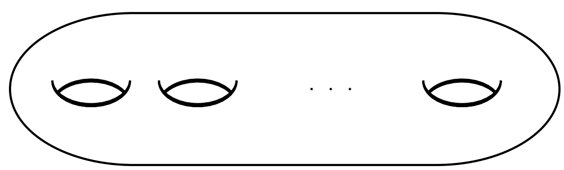
7. Algebraic topology of surfaces

The exercises in this section are concerned with aspects of the fundamental groups of closed oriented surfaces, and of higher-dimensional manifolds built out of surfaces.

Definition 7.1. The surface of genus g, denoted Σ_g, is the set of points in $\mathbb{R}^3$ at a distance ε from the following graph X_g, embedded as shown in $\mathbb{R}^3$:



g circles



g holes

One approach to computing $\pi_1(\Sigma_g)$ is to proceed via *deformation retractions.*

Definition 7.2. Let $A \subset X$ be a pair of spaces. A map $f : X \times I \to X$ is a *deformation retraction onto* A if the following is satisfied:

(1) $f(\cdot, 0) = id$,
(2) $f(x, 1) \in A$ for all $x \in X$,
(3) $f(a, t) = a$ for all $a \in A, t \in I$.

Theorem 7.3. *If f is a deformation retraction of X onto A, then $f(\cdot, 1) : X \to A$ induces an isomorphism $f_{1,*} : \pi_1 X \to \pi_1 A$.*

➪**Exercises 7.4.**
(1) **(B)** Let Σ_1^1 denote Σ_1 after deleting an open disk. Compute $\pi_1 \Sigma_1^1$. You may assume the result of Section 5, Exercise (2). [Hint: The idea is to find a deformation retraction of Σ_1^1 onto the graph X.] (Definitions)
(2) **(B)** Now let Σ_1^2 denote Σ_1 after deleting two open disks that have disjoint closures. Compute $\pi_1 \Sigma_1^2$.

(3) **(B)** Use the previous exercises to compute $\pi_1\Sigma_g$ (here of course we are dealing with *closed* surfaces!). [Hint: Use induction.] (van Kampen)

(4) **(B)** Draw a picture of Σ_2. Find a point $p \in \Sigma_2$ and elements a, b, c, d of $\pi_1(\Sigma_2, p)$ for which $\pi_1\Sigma_2$ has the following presentation:

$$\pi_1\Sigma_2 = \langle a, b, c, d \mid [a, b][c, d] = 1\rangle.$$

(Remember that $[x, y] = xyx^{-1}y^{-1}$ is the *commutator* of x, y).

(5) Is the map $f : \pi_1\Sigma_2 \to \mathbb{Z}/2$ taking a to 1 and all other generators to 0 actually well-defined? If so, determine what the corresponding cover Y of Σ_2 looks like. Is it another surface? What is its genus? How does the quotient $\mathbb{Z}/2$ act on Y? Can you find a presentation for $\pi_1 Y$? (Galois correspondence)

(6) Check that the map $\alpha : \pi_1\Sigma_2 \to \pi_1\Sigma_2$ taking a to ab and fixing all other generators is a well-defined automorphism of $\pi_1\Sigma_2$.

(7) Find a homeomorphism $f : \Sigma_2 \to \Sigma_2$ which fixes your choice of $p \in \Sigma_2$, for which f_* is given by α.

(8) The *mapping torus* of f is the following 3-manifold obtained by a quotient construction:

$$M_f = \Sigma_2 \times [0, 1]/\{(x, 1) = (f(x), 0)\}.$$

Compute $\pi_1 M_f$. (van Kampen)

8. Further Exercises

(1) Find a degree two covering map $f : \Sigma_3 \to \Sigma_2$. Let $F : \Sigma_3 \to \Sigma_3 \times \Sigma_2$ be the *graph* of f. What map does F induce on π_1?

(2) Suppose Σ is a compact orientable surface and $\gamma : S^1 \to \Sigma$ is a simple closed curve that is not homotopic to a point. Suppose further that γ is non-separating, so $\Sigma \smallsetminus \operatorname{Im}\gamma$ has one path-component Σ_0, and two-sided (that is, γ is not the core of a Möbius band). Then show that

$$\pi_1(\Sigma) \cong \pi_1(\Sigma_0) *_{\langle\gamma\rangle} .$$

(3) Let $f : X \to X$ be a homeomorphism from a CW-complex X to itself. Calculate $\pi_1(M_f)$ in terms of $\pi_1(X)$ and f_*.

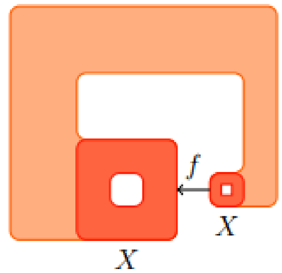
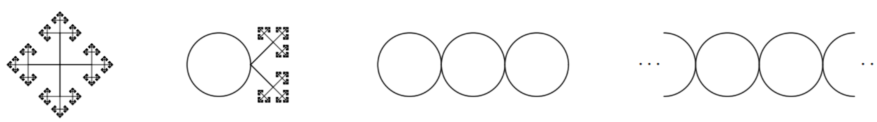


FIGURE 4. Mapping Torus M_f

(4) For each of the following covering spaces of $X = S^1 \vee S^1$, identify the corresponding subgroup of $\pi_1(X)$, and say whether the following covers are regular (i.e. normal) or not.



(5) Let Σ_g be a surface of genus $g \geq 0$. Let $\Delta \subset \Sigma_g \times \Sigma_g$ be the image of the diagonal embedding $x \mapsto (x, x)$. Let X be the complement of Δ in $\Sigma_g \times \Sigma_g$. Compute $\pi_1(X)$.
(6) Let Y be the complement of the diagonal $\{(x, x) : x \in \mathbb{T}^3\}$ in $\mathbb{T}^3 \times \mathbb{T}^3$ where $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. What is $\pi_1(Y)$?
(7) We know that there is no way to define a contnuous square root map on the entire complex plane. More generally, prove that we cannot always find a square root of a complex valued function on a given topological space. In particular show the following:
   (a) Let X be a topological space and let $f : X \to \mathbb{C}$ be a function which is never 0. Show that there exists a degree two covering space $p : \tilde{X} \to X$ such that p^*f has a square root i.e. $\exists \tilde{f} : \tilde{X} \to \mathbb{C}$ such that $\tilde{f}(x)^2 = f(p(x))$.
   (b) Show that f has a square root *if and only if* $p : \tilde{X} \to X$ is a trivial covering space i.e. isomorphic to the covering space $X \coprod X \to X$.
   [Hint: Use path lifting criterion]
(8) Using the fact that trefoil knot is a Torus knot, compute $\pi_1(\mathbb{R}^3 - K)$ where K is the Trefoil knot. Do it for any torus knot in general.

FIGURE 5. Trefoil Knot

(9) What is $\pi_1(SL(2,\mathbb{R}))$ regarded as a subspace of $\mathbb{R}^4$?
[Hint: Show that $SL(2,\mathbb{R})$ is homeomorphic to $\{$Group of upper triangular matrices with positive entries in the diagonal$\}\times\{$Rotation matrices$\}$]

(10) Given an arbitrary group G, can you construct a space such that its fundamental group is G?

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