INMO Training Camp Problem Sheet

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January 10, 2012

1 Some Problems

1. Let ABC be a triangle in which $\angle BAC = 60°$. Let P (similarly Q) be the point of intersection of the bisector of $\angle ABC$(similarly of $\angle ACB$) and the side AC(similarly AB). Let r_1 and r_2 be the in-radii of the triangles ABC and APQ, respectively. Determine the circumradius of APQ in terms of r_1 and r_2.

2. Let P be a point inside a triangle ABC such that $\angle PAB = \angle PBC = \angle PCA$ Suppose AP, BP, CP meet the circumcircles of triangles PBC, PCA, PAB at X, Y, Z respectively ($\neq P$) . Prove that $[XBC] + [YCA] + [ZAB] \geq 3[ABC]$

3. Given a triangle ABC, the internal and external bisectors of angle A intersect BC at points D and E respectively. Let F be the point (different from A) where line AC intersects the circle w with diameter DE. Finally, draw the tangent at A to the circumcircle of triangle ABF, and let it hit w at A and G. Prove that $AF = AG$.

4. Angle bisectorsAA_1, BB_1, CC_1 in triangle $\triangle ABC$ intersect at incenter I. Line B_1C_1 intersects circumcircle of triangle $\triangle ABC$ at M and N. R_1 is radius length of circle circumscribed to the triangle $\triangle MIN$, and R radius length of circle circumscribed to the triangle $\triangle ABC$ Prove that $R_1 = 2R$.

5. D and E are points on side BC of $\triangle ABC$ such that $\dfrac{BD}{DC} \cdot \dfrac{BE}{EC} = \dfrac{AB^2}{AC^2}$. Show that $\angle DAB = \angle EAC$.

6. Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of $\triangle ABC$. Prove that D, O, H are collinear.

7. Let E, F be points on sides AC, AB of triangle ABC. The circumcircle of $\triangle AFE$ meets the circumcircles of $\triangle BEC$ and $\triangle BFC$ in M and N respectively. If BM meets CN in T, prove that $AT \| BC$.

8. Let $p(x)$ and $q(x)$ be polynomials such that$[p(x^2 + 1)] = [q(x^2 + 1)]$. Prove that $p(x) = q(x)$. Here [.] denotes greatest integer function.

9. Solve the following equation for $x \in \left(0, \frac{\pi}{2}\right)$:

$$\frac{2\sqrt{x}}{\pi} + \sqrt{\sin x} + \sqrt{\tan x} = \frac{1}{2\sqrt{x}} + \sqrt{\cot x} + \sqrt{\cos x}.$$

10. Find all polynomials $p(x)$ with $p(0) = 0$ such that for the strictly increasing function $f : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+$ the following equality holds: $2p(f(x)) = f(p(x)) + f(x)$.

11. Find all polynomials $p(x)$ and $q(x)$ such that:

$$p^2(x) + q^2(x) = (3x - x^3)p(x)q(x), \forall x \in (0, \sqrt{3})$$

12. Find all polynomials $p(x)$ such that, for all non zero real numbers x, y, z such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ we have:

$$\frac{1}{p(x)} + \frac{1}{p(y)} = \frac{1}{p(z)}$$

13. The polynomial $p(x)$ has positive coefficients and degree n. If the sum of the reciprocals of its coefficients equals 1, prove that

$$\sqrt{p(4)} + 1 \geq 2^{n+1}$$

14. Find all polynomials $P(x)$ with real coefficients that satisfy the equality $P(a-b)+P(b-c)+P(c-a) = 2P(a+b+c)$ for all triples a, b, c of real numbers such that $ab+bc+ca=0$.

15. Let P be any point inside $\triangle ABC$ with circumcircle (O). AP, BP, CP meet (O) again in X, Y, Z respectively. Q is any point on (O). Lines QX, QY, QZ meet the sides BC, CA, AB in K, L, M respectively. Show that K, M, P, L are collinear.

16. (□)Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

17. Let n be natural and $1 = d_1 < d_2 < ... < d_k = n$ be the positive dividers of n. Find all the n such that $2n = d_5^2 + d_6^2 - 1$.

18. The three roots of $P(x) = x^3 - 2x^2 - x + 1$ are $a > b > c \in \mathbb{R}$. Find the value of $a^2b + b^2c + c^2a$.

19. Given a quadrilateral $ABCD$ which is inscribed in (O) such that $AC \perp BD$. The tangents of (O) through A, B, C, D intersect each other and make a circumscribed quadrilateral $XYZT$. XZ intersects YT at P. Prove that the incenters of 8 triangles $XPY, YPZ, ZPT, TPX, XYZ, YZT, ZTX, TXY$ are concyclic.

20. Let a, b, c be given positive integers. Prove that there exists some positive integer N such that $a \mid Nbc + b + c$, $b \mid Nca + c + a$, $c \mid Nab + a + b$ if and only if, denoting $d = gcd(a, b, c)$ and $a = dx, b = dy, c = dz$, the positive integers x, y, z are pairwise coprime, and also $gcd(d, xyz) \mid x + y + z$.

21. Find all Polynomials $P(X) \in \mathbb{R}[X]$ which satisfy $P(sin(x)) = P(cos(x)) \forall x \in \mathbb{R}$.

22.
    - If z_1, z_2, z_3 denote the vertices of a triangle $\triangle ABC$ with center at origin. What is the Nine-Point Centre?
    - Take a quadrilateral $ABCD$ inscribed in a circle. Prove that the nine-point centers of the triangles $ABC, ABD, BCDandCDA$ are concyclic.(lie on w , let.)
    - Define the center of the above circle w to be the Nine-Point Center of the quadrilateral $ABCD$. Can you generalise the concept to a general polygon inscribed in a circle?

2 Some Theory (Harmonic Division):

Let AB be a line segment. Points P and Q lying on AB are said to divide AB harmonically if $\frac{AP}{PB} = \frac{AQ}{BQ} = -\frac{AQ}{QB}$ .So, P and Q divide AB in the same ratio one internally and one externally. If the points P, Q divide AB harmonically, then A, B also divide PQ harmonically (Check!). We shall say that $(A, B; P, Q)$ is a harmonic division whenever P and Q divide AB harmonically.

The following theorems have been left as exercise:

- Let p and q be two distinct lines in a plane and P is point in the plane not on any of these lines. A, C, B, D are points on p such that $(A, B; C, D)$ is a harmonic division. PA, PB, PC, PD meet the line q in K, L, M, N respectively. Show that $(K, M; L, N)$ is a harmonic division. (Use Menelaus Theorem to prove this.)
- Suppose A, C, B, D are points on a line such that $(A, B; C, D)$. P is a point in the plane such that $\angle APB = 90°$. Then PB bisects $\angle PBD$.

1. AD is the altitude of $\triangle ABC$ with D on BC. P is any point on AD . The lines BP, CP meet the sides AC, AB in E, F respectively. Show that AD bisects $\angle EDF$. [Hint: Let DF meet BE in K. Show that $(B, P; K, E)$ is a harmonic division.Now apply Theorem 2.]

2. Let A, B, C, D be points in a line such that $(A, C; B, D)$ is a harmonic division. If O is the midpoint of AB, show that $OB^2 = OC \cdot OD$.

3. Suppose AD, BE, CF are three concurrent lines in a $\triangle ABC$ with D, E, F lying on BC, CA, AB respectively. If EF meets BC in T, show that $(B, C; D, T)$ is a harmonic division.

4. Let $ABCD$ be a convex quadrilateral. AD meets BC in K and the diagonals AC, BD meet in E. If KE meets AB and CD in P and Q respectively, show that $(K, E; Q, P)$ is a harmonic division.