KINMOTC 2011-Problem Sheet

Subhadip Chowdhury
bmat0917@isibang.ac.in

January 7, 2012

The $\diamond$ marked problems are easier. But other than that, the following are *NOT* according to order of difficulty

1. Given a scalene triangle ABC. Let A', B', C' be the points where the internal bisectors of the angles CAB, ABC, BCA meet the sides BC, CA, AB, respectively. Let the line BC meet the perpendicular bisector of AA' at A''. Let the line CA meet the perpendicular bisector of BB' at B'. Let the line AB meet the perpendicular bisector of CC' at C''. Prove that A'', B'' and C'' are collinear.

2. In triangle ABC, M is midpoint of AC, and D is a point on BC such that $DB = DM$. We know that $2BC^2 - AC^2 = AB.AC$. Prove that
$$BD.DC = \frac{AC^2.AB}{2(AB + AC)}$$

3. $\diamond$ A convex quadrilateral is circumscribed about a circle. 4 circles are inscribed in the quadrilateral in such a way that each of them touches two consequtive sides of the quadrilateral and two neighbouring circles. Prove that at least two of the 4 circles have same radii.

4. A convex quadrilateral $ABCD$ is circumscribed about a circle with centre P. A straight line through P intersects AB and CD at X and Y respectively. If $PX = PY$,prove that
$$AX.DY = BX.CY$$

5. (a) $\diamond$ If z_1, z_2, z_3 denote the vertices of a triangle ABC with center at origin. What is the Nine-Point Centre?
   (b) Take a quadrilateral $ABCD$ inscribed in a circle. Prove that the nine-point centres of the triangles ABC, ABD, BCD and CDA are concyclic.(lie on ω , let.)
   (c) Define the center of the above circle ω to be the Nine-Point Center of the quadrilateral $ABCD$. Can you generalise the concept to a general polygon inscribed in a circle?

6. At the corners of any equilateral triangle $A_1B_1C_1$ let there be hinged three equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ of any sizes and positions. Then will the midpoints of each of the sets of three segments (B_1C_3, C_1B_2, B_3C_2), (A_2B_1, B_3A_1, A_3B_2) ,(A_1C_2, C_1A_3, A_2C_3) be the corners of an equilateral triangle?

7. The whole plane is coloured using 2 colours.
   (a) $\diamond$ Show that one can always get 2 points at $d(\in \mathbf{R}, d > 0)$ distance which have the same colour.
   (b) $\diamond$ Prove that there exists a colour which contains points at every mutual distance.
   (c) Show that a monochromatic equilateral triangle is present. In fact show that it is possible to get a monochromatic equilateral triangle of either side 1 or side $\sqrt{3}$. Also demonstrate a colouring such that one has no monochromatic equilateral triangle of side 1.
   (d) Prove that some rectangle has its vertices all the same color.

8. Let ABC be an equilateral triangle (i. e., a triangle which satisfies $BC = CA = AB$). Let M be a point on the side BC, let N be a point on the side CA, and let P be a point on the side AB, such that $S(ANP) = S(BPM) = S(CMN)$, where $S(XYZ)$ denotes the area of a triangle XYZ. Prove that $\triangle ANP \cong \triangle BPM \cong \triangle CMN$.

9. Let P be an interior point of a circle and $A_1, A_2, \cdots, A_{10}$ be points on the circle such that $\angle A_1PA_2 = \angle A_2PA_3 = ... = \angle A_{10}PA_1 = \frac{\pi}{5}$. Prove that $PA_1 + PA_3 + PA_5 + PA_7 + PA_9 = PA_2 + PA_4 + PA_6 + PA_8 + PA_{10}$.

10. If $\angle ABC$ in $\triangle ABC$ is equal to $\frac{\pi}{3}$, prove that A, N(Nine Point Center) and I (Incenter) are collinear.

11. Let $ABCD$ be a trapezeum with $AB||DC$ and $AB > CD$. E and F are points on AB and DC respectively such that $\frac{AE}{EB} = \frac{DF}{FC}$. Let K and L be points on the segment EF such that $\angle BKA = \angle BCD$ and $\angle DLC = \angle ABC$. Show that K, L, B, C are concyclic.

12. In a circle with centre at O and diameter AB, two chords BD and AC intersect at E. F is a point on AB such that $EF \perp AB$. If CF intersects BD in G. If $DE = 5$ and $EG = 3$, determine BG.

13. Let M, Y, X be mid-points of the three sides AB, BC, CA of $\triangle ABC$. Let P be the foot of perpendicular from M to the internal angle bisector of $\angle BAC$. If $PX = PY$, find $\angle BAC$.

14. $\diamond$ Show that there are no regular polygons with more than 4 sides inscribed in an ellipse (which is not a circle).

15. $\diamond$ Let AA', BB' and CC' be three non-coplanar chords of a sphere and let them all pass through a common point P inside the sphere. There is a (unique) sphere S_1 passing through A, B, C, P and a (unique) sphere S_2 passing through A', B', C', P. If S_1 and S_2 are externally tangent at P, then prove that $AA' = BB' = CC'$.

16. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b , there exists a non-degenerate triangle with sides of lengths $a, f(b)$ and $f(b + f(a) - 1)$.

17. $\diamond$ Let $a \neq b$ and $a, b \in \mathbf{R}$ such that $(x^2 + 20ax + 10b)(x^2 + 20bx + 10a) = 0$ has no roots for x. Prove that $20(b - a)$ is not an integer.

18. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ . Prove that $OP = OQ$.

19. Let the function $f : R^+ \to R^+$ satisfy the following condition: $f(x)f(y) = f(x + yf(x))$ .Find all of such functions.

20. A polynomial$P(x)$ with real coefficients and of degree $n \geq 3$ has n real roots $x_1 < x_2 < \cdots < x_n$ such that $x_2 - x_1 < x_3 - x_2 < \cdots < x_n - x_{n-1}$ Prove that the maximum value of $|P(x)|$on the interval $[x_1, x_n]$ is attained in the interval $[x_{n-1}, x_n]$.

21. $\diamond$ Two polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ have real coefficients, and I is an interval on the real line of length greater than 2. Suppose $P(x)$ and $Q(x)$ take negative values on I, and they take non-negative values outside I. Prove that there exists a real number x_0 such that $P(x_0) < Q(x_0)$.

22. Call a positive integer *good* if either $N = 1$ or N can be written as product of even number of prime numbers, not necessarily distinct. Let $P(x) = (x - a)(x - b)$, where a, b are positive integers.

    (a) $\diamond$ Show that there exist distinct positive integers a, b such that $P(1), P(2), \cdots, P(2010)$ are all *good* numbers.

    (b) Suppose a, b are such that $P(n)$ is a *good* number for all positive integers n. Prove that $a = b$.

23. Find all Polynomials $P(X) \in \mathbf{R}[X]$ which satisfy $P(sin(x)) = P(cos(x)) \forall x \in \mathbf{R}$.