KINMOTC-Problem Sheet

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1. (Postal-09)Let P be an interior point of a circle and $A_1, A_2, \cdots, A_{10}$ be points on the circle such that $\angle A_1PA_2 = \angle A_2PA_3 = ... = \angle A_{10}PA_1 = \frac{\pi}{5}$. Prove that $PA_1 + PA_3 + PA_5 + PA_7 + PA_9 = PA_2 + PA_4 + PA_6 + PA_8 + PA_{10}$.

2. (IMO 72)Prove that for each $n \geq 4$ every cyclic quadrilateral can be decomposed into n cyclic quadrilateral.

3. Let AL be the angle bisector of $\angle BAC$ in $\triangle ABC, (L \in BC)$.Let N be the point where AL again intersects the circumcircle of $\triangle ABC$.Let M and N be the feet of perpendiculars from L to AB and AC. Prove that area of $\triangle ABC$ and quadrilateral $AKNM$ are equal.

4. Let ABC be a triangle and O its circumcenter. Lines AB and AC meet the circumcircle of OBC again in $B_1 \neq B$ and $C_1 \neq C$, respectively, lines BA amd BC meet the circumcircle of OAC again in $A_2 \neq A$ and $C_2 \neq C$, respectively, and lines CA and CB meet the circumcircle of OAB in $A_3 \neq A$ and $B_3 \neq B$, respectively. Prove that lines A_2A_3, B_1B_3 and C_1C_2 have a common point.

5. (IMOSL 88) Let ABC be an acute-angled triangle. Three lines L_A, L_B, and L_C are constructed through the vertices A, B, and C respectively according to the following prescription: Let H be the foot of the altitude drawn from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N respectively, where M and N are distinct from A; then L_A is the line through A perpendicular to MN . The lines L_B and L_C are constructed similarly. Prove that L_A, L_B , and L_C are concurrent.

6. Let M be the midpoint of the side AC of a triangle ABC and let H be the footpoint of the altitude from B. Let P and Q be orthogonal projections of A and C on the bisector of the angle B. Prove that the four points H,P,M and Q lie on the same circle.

7. Let ABC be a triangle with $AB = AC$. Also, let $D \in [BC]$ be a point such that $BC > BD > DC > 0$, and let $\mathcal{C}_1, \mathcal{C}_2$ be the circumcircles of the triangles ABD and ADC respectively. Let BB' and CC' be diameters in the two circles, and let M be the midpoint of $B'C'$. Prove that the area of the triangle MBC is constant (i.e. it does not depend on the choice of the point D).

8. Let a circle intersect the three sides BC, CA, AB of a triangle ABC at $D_1, D_2; E_1, E_2; F_1, F_2$ respectively. Let $L = D_1E_1 \cap D_2F_2$. M and N are similarly defined. Prove that AL, BM, CN are concurrent.

9. Let the incircle of triangle ABC touch sides BC,CA,AB at D,E,F respectively. Let AD intersect the incircle at P($\neq D$). Let the perpendicular to AD at P and EF intersect at Q. Let DE and DF intersect AQ at X and Y respectively.Prove that AX=AY.

10. (IMOSL 91)In the triangle ABC , $\angle A = \frac{\pi}{3}$, a parallel IF to AC is drawn through the incenter I of the triangle, where F lies on the side AB. The point P on the side BC is such that $3BP = BC$.Show that $\angle BFP = \frac{\angle B}{2}$.

11. Let ABCDEF be a cyclic convex hexagon with AB=CD=EF and $AD \cap BE \cap CF = Q$.Let $P = CE \cap AD$.Prove that $\frac{AC^2}{FD^2} = \frac{PC}{PE}$.

12. (IMO 93) On an infinte chessboard, a solitaire game is played as follows: At the start, we have n^2 pieces occupying n^2 squares that form a square of side n. The only allowed move is a jump horizontally or vertically over an occupied square to an unoccupied one, and the piece that has been jumped over is removed. For what positive integers n can the game end with only one piece remaining on the board?

13. Prove that in any convex $n - gon$ there exist 3 consequtive vertices whose circumcircle covers the whole $n - gon$.

14. Suppose a 8×8 chessboard is covered using I-trominoes. If we are able to put 21 trominoes on it, determine all possible position for the last empty square.

15. (USAMO)Determine (with proof) whether there is a subset X of the non-negative integers with the following property: for any natural n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

16. Suppose a fox is sitting inside a $2n - gon$ but not on any diagonal. 2n hunters sitting on the vertices shoot at the fox simultaneously but the fox ducks in time and the bullet hits one of the sides of the $2n - gon$. Prove that there is at least one side of the $2n - gon$ which is not hit by any bullet.

17. At the corners of any equilateral triangle $A_1B_1C_1$ let there be hinged three equilateral triangles $A_1A_2A_3, B_1B_2B_3, C_1C_2C_3$ of any sizes and positions. Then will the midpoints of each of the sets of three segments $(B_1C_3, C_1B_2, B_3C_2), (A_2B_1, B_3A_1, A_3B_2), (A_1C_2, C_1A_3, A_2C_3)$ be the corners of an equilateral triangle?