Stable Commutator Length and Quasimorphisms

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§0. CONTENTS

§1. INTRODUCTION

Like virtually every other category, in the Topological category as well, it is important to be able to construct and classify surfaces of least complexity mapping to a given space. The problem amounts to minimizing the genus of a surface mapping to some space; often considered subject to further constraints. To understand why it is so, note that the second homology of a space describes the failure of $2-$cycles to bound $3-$simplices. Intuitively, the rank of H_2 of a space counts the number of $2-$dimensional "holes" the space has. Now consider a topological space X and let $\alpha \in H_2(X)$ be an integral homology class, represented by an integral 2-cycle A. By definition there is an expression of the form $A = \sum_i n_i\sigma_i$ where $n_i \in \mathbb{Z}$ and σ_i's are singular $2-$simplices. Without loss of generality, allowing repetition of σ_i, we may assume each of the n_i is ± 1. Since $\partial A = 0$, each face e of some σ_i appears an even number of times with opposite signs in $\sum_i n_i \partial\sigma_i$. Thus we may choose a pairing of the faces of σ_i so that each of them contribute 0 in the sum.

We build a simplicial $2-$complex K by taking one 2-simplex for each σ_i, and gluing the edges according to this pairing. We thus obtain an oriented surface S and an induced continuous map $f_A : S \to X$ such that by construction

$$(f_A)_*([S]) = [A] = \alpha$$

where $[S] \in H_2(S)$ is the fundamental class. Thus, elements of $H_2(X)$ are represented by maps of closed oriented surfaces into X.

For many applications, it is necessary to relativize the problem: given a space X and a (homologically trivial) loop γ in X, we want to find the surface of least complexity (perhaps subject to further constraints) mapping to X in such a way that γ is the boundary. On the algebraic side, the relevant homological tool to describe complexity in this context is *stable commutator length*, which is the main topic of next section.

§2. STABLE COMMUTATOR LENGTH

◇ 2.1 Group theoretic Definition

Definition 2.1. Let G be a group, and $a \in [G, G]$. The *commutator length* of a, denoted $cl(a)$, is the least number of commutators in G whose product is equal to a. By convention we write $cl(a) = \infty$ for $a \notin [G, G]$.

Definition 2.2. For $a \in [G, G]$, the *stable commutator length*, denoted by $\mathrm{scl}(a)$, is the following limit

$$\mathrm{scl}(a) = \lim_{n\to\infty} \frac{cl(a^n)}{n}$$

Note that since the function $n \mapsto cl(a^n)$ is subadditive, the limit exists. It is easy to see that to investigate properties of scl, it is enough to restrict to the *countable* subgroup generated by the element.

Definition 2.3. Let G be a group, and $g_1, g_2, \ldots, g_m$ elements of G whose product is in [G, G]. Define

$$cl\left(g_1 + \ldots + g_m\right) = \inf_{h_i \in G} cl\left(g_1 h_1 g_2 h_1^{-1} \ldots h_{m-1} g_m h_{m-1}^{-1}\right)$$

and

$$\mathrm{scl}\left(g_1 + \ldots + g_m\right) = \lim_{n\to\infty} \frac{cl\left(g_1^n + \ldots + g_m^n\right)}{n}$$

2.1.1 | Examples

- Let $F_2 = \langle a, b \rangle$. Then in general,

  $$cl_{F_2}([a, b]^n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

  Hence

  $$\mathrm{scl}_{F_2}\left([a, b]\right) = \frac{1}{2}$$

- For a knot $\gamma : S^1 \hookrightarrow S^3$, $cl(\gamma) = g(\gamma)$, the genus of γ where γ is regarded as an element in $\pi_1(S^3 \backslash N(\gamma))$ and $N(\gamma)$ is an open neighborhood of γ.

- scl is identically zero on $[G, G]$ for many important class of groups G, e.g.

  - torsion groups
  - solvable groups, and more generally, amenable groups
  - $SL(n, \mathbb{Z})$ for $n \geq 3$, and many other lattices (uniform and nonuniform) in higher rank Lie groups

- In particular, as we will see, scl vanishes identically on $[G, G]$ if and only if every homogeneous quasimorphism on G is a homomorphism.

◇ 2.2 Geometric Characterization

The properties of scl are often more clear from its geometric characterization.

2.2.1 | Fundamental group and Commutators

Let $\gamma \in \pi_1(X)$ be a conjugacy class represented by a loop l_γ in a topological space X. If γ has a representative in the commutator subgroup $[\pi_1(X), \pi_1(X)]$, then we can find $\alpha_i, \beta_i \in \pi_1(X)$ such that

$$[\alpha_1, \beta_1] \dots [\alpha_g, \beta_g] = \gamma \in \pi_1(X)$$

Let S be a genus g surface with one boundary component, which can be obtained from a $(4g+1)-$gon P by identifying sides in pairs. We use the 'standard' representation of

$$\pi_1(S) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] \rangle$$

to define maps from the edges of P to loops in X to get that loops corresponding to elements of $[\pi_1(X), \pi_1(X)]$ are boundaries of maps of oriented surfaces into X.

Given a group G, consider a topological space such that $\pi_1(X) = G$. It follows that the commutator length of an element $a \in G$ is the least genus of a surface with one boundary component mapping to X in such a way that the boundary represents the free homotopy class of a loop γ corresponding to $a \in \pi_1(X)$. Then the stable commutator length can be calculated by considering genus of surfaces whose boundary wraps around γ multiple times.

2.2.2 | scl as an intrinsic geometric property

Definition 2.4. Given a compact oriented surface S, let $-\chi^-(S)$ denote the sum of the $\max\{-\chi(.), 0\}$ over the connected components of S.

Let $g_1, \dots, g_m \in G$ be given so that the product is trivial in $H_1(G; \mathbb{Q})$. Let X be a space with $\pi_1(X) = G$ and let $\gamma_i : S^1 \to X$ be nontrivial free homotopy classes of loops representing the conjugacy class of g_i. For a compact oriented surface S, we say that a map $f : S \to X$ is *admissible* of degree $n(S)$ if the following diagram commutes:

$$\begin{tikzcd}
\partial S \arrow[r, "i"] \arrow[d, "\partial f"'] & S \arrow[d, "f"] \\
\coprod_i S^1 \arrow[r, "\coprod_i \gamma_i"] & X
\end{tikzcd}$$

and

$$\partial f_*[\partial S] = n(S) \left[\coprod_i S^1 \right].$$

We can now give an intrinsic geometric definition of scl.

Proposition 2.5. *With notations as above,*

$$\mathrm{scl}(g_1 + \dots + g_m) = \inf_S \frac{-\chi^-(S)}{2n(S)}$$

where the infimum is taken over all admissible maps as above.

Definition 2.6. The chain $g_1 + \cdots + g_m$ is said to *rationally bound* a surface S is $f : S \to X$ is admissible. Such a surface is *extremal* if every component has negative Euler characteristic and it realizes the infimum of $\frac{-\chi^-(S)}{2n(S)}$.

Extremal surfaces are π_1-injective. The following proposition tells us what kind of admissible surfaces are enough to consider. By passing to a suitable cover and gluing together boundary components mapping with opposite degree to the same circle, we can prove,

Proposition 2.7. *Let S be connected with $\chi(S) < 0$, and let $f : S, \partial S \to X, \gamma$ be admissible. Then there is an admissible map $f' : S', \partial S' \to X, \gamma$ with positive degree on every component of $\partial S'$ such that*

$$\frac{-\chi^-(S')}{2n(S')} \leq \frac{-\chi^-(S)}{2n(S)}$$

2.2.3 | Examples

The following examples highlights the usefulness of the geometric characterization in explicit calculation of scl.

- We know that there is no map of nontrivial degree from a surface of genus g to a surface of genus $h > g$. It follows from the fact that if you have a map from lower genus to higher, there is an element a in the kernel of the induced map on first cohomology (say over $\mathbb{Q}$) and so the fundamental class $a \smile b$ (for some b, by Poincaré duality) goes to zero (by naturality of cup product). Thus given a surface $X = \Sigma_g$ of genus $g \geq 1$ the infimum of $\frac{-\chi^-(S)}{2n(S)}$ is equal to $-\chi^-(\Sigma_g)/2$.
- Suppose G obeys a law. Then the stable commutator length vanishes identically on $[G, G]$.

◇ 2.3 Functional Analytic Characterization

Aside from above two characterizations, we can also give a functional theoretic characterization of scl. By *Bavard duality theorem* 2.13 we can define scl dually in terms of certain kind of functions on groups, namely *quasimorphisms* and *bounded cohomology* of the group.

2.3.1 | Quasimorphisms

Definition 2.8. Let G be a group. A *quasimorphism* is a function $\varphi : G \to \mathbb{R}$ for which there exists a least constant $D(\varphi) \geq 0$ such that

$$|\varphi(ab) - \varphi(a) - \varphi(b)| \leq D(\varphi)$$

for all $a, b \in G$. $D(\varphi)$ is called the *defect* of φ.

Clearly any bounded function is trivially a quasimorphism.

■ Homogenization:

Definition 2.9. A quasimorphism is *homogeneous* if $\varphi(a^n) = n\varphi(a)$ for all $a \in G, n \in \mathbb{Z}$. We denote the vector space of all homogeneous quasimorphisms on G by $Q(G)$.

Lemma 2.10. *For a quasimorphism φ on G, the limit*

$$\overline{\varphi}(a) := \lim_{n\to\infty} \frac{\varphi(a^n)}{n}$$

exists and defines a homogeneous quasimorphism $\overline{\varphi}$.

Proof. (Sketch) Note that $\varphi(a^{2^i})2^{-i}$ is a Cauchy sequence. Then to prove homogeneity, use triangle inequality and induction. □

■ Commutator estimates: If φ is homogeneous then

$$\left|\varphi(aba^{-1}) - \varphi(b)\right| = \frac{1}{n}\left|\varphi(ab^n a^{-1}) - \varphi(b^n)\right| \leq \frac{2D(\varphi)}{n}$$

which implies φ is a class function. Thus

$$D(\varphi) \geq |\varphi([a, b]) - \varphi(aba^{-1}) - \varphi(b^{-1})| = |\varphi([a, b]) - \varphi(b) + \varphi(b)| = |\varphi([a, b])| \tag{2.3.1.1}$$

It turns out that

Lemma 2.11 (Bavard,[3])**.** *Let φ be a homogeneous quasimorphism on G. Then there is an equality*

$$\sup_{a,b} |\varphi([a, b])| = D(\varphi)$$

2.3.2 | Bounded Cohomology

Definition 2.12. The *bar complex* $C_*(G)$ of a group G is the complex generated in dimension n by $n-$tuples $(g_1, \ldots, g_n)$ with $g_i \in G$ and with boundary map ∂ defined by the formula

$$\partial(g_1, \ldots, g_n) = (g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n (g_1, \ldots, g_{n-1}).$$

For a coefficient group R, we let $C^*(G; R)$ denote the dual cochain complex $\hom(C_*(G), R)$, and let δ denote adjoint of ∂. The homology groups of $C^*(G; R)$ are called the group cohomology of G with coefficients in R, and are denoted by $H^*(G; R)$.

If R is a subgroup of $\mathbb{R}$, a cochain $\alpha \in C^n(G)$ is *bounded* if $\sup |\alpha(g_1, \ldots, g_n)|$ is finite, where the supremum is taken over all generators. This supremum is called the norm of α, denoted by $\|\alpha\|_\infty$. The set of all bounded cochains forms a subcomplex $C_b^*(G)$ of $C^*(G)$, and its homology is called the *bounded cohomology* $H_b^*(G)$.

A real valued function φ on G may be thought of as a $1-$cochain, i.e. an element of $C^1(G; \mathbb{R})$. The coboundary δ of such a function is defined by the formula

$$\delta\varphi(a, b) = \varphi(a) + \varphi(b) - \varphi(ab)$$

At the level of norms, there is an equality $\|\delta\varphi\|_\infty = D(\varphi)$. Hence the coboundary of a quasimorphism is a bounded $2-$cocycle. Two $1-$cochains have the same image in H_b^2 under δ iff they differ by a bounded function. But if two quasimorphisms differ by a bounded function, under homogenization they give the same quasimorphism. It follows that we have an exact sequence

$$0 \to H^1(G; \mathbb{R}) \to Q(G) \xrightarrow{\delta} H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R}) \tag{2.3.2.1}$$

Let $B_n(G; \mathbb{R})$ denote the subspace of real group $n-$boundaries. Then using definition 2.3, we can think of scl as a function on the set of integral group $1-$boundaries. This function being linear on rays and subadditive, admits a unique continuous linear extension to $B_1(G)$. Let $H(G)$ denote the subspace of $B_1(G)$ spanned by chains of the form $g - hgh^{-1}$ and $g^n - ng$ for $g, h \in G$ and $n \in \mathbb{Z}$. Thus by construction, scl descends to a pseudo-norm on $B_H^1 := B_1(G)/H(G)$. Using the following theorem, we can show that scl is a genuine norm on $B_1^H(G)$ for a hyperbolic group G.

2.3.3 | Bavard's Duality Theorem

We relate quasimorphisms and stable commutator length via the following theorem:

Theorem 2.13 (Bavard's Duality Theorem,[3])**.** *Let G be a group. Then for any $\sum_i t_i a_i \in B_1(G)$, we have an equality*

$$\operatorname{scl}\left(\sum_i t_i a_i\right) = \frac{1}{2}\left(\sup_{\varphi \in Q/H^1} \frac{\sum_i t_i \varphi(a_i)}{D(\varphi)}\right)$$

Proof. (Sketch) We can prove that the dual of $B_1(G)$ with the Gersten boundary norm $\|.\|_b$ is $\widehat{Q}/H^1$ with $D(.)$ norm. Also for any linear chain $\sum_i t_i a_i \in B_1$, $\operatorname{scl}(\sum_i t_i a_i) = \frac{1}{4}\operatorname{fill}(\sum_i t_i a_i)$, where fill(.) is the homogenization of $\|.\|_b$. The result then follows directly by an application of Hahn-Banach theorem. □

We see that scl is a kind of relative Gromov-Thurston norm. In fact they are related and the unit norm balls satisfy similar properties.

◇ 2.4 The Rationality Theorem for scl in Free Groups

We are going to see that the unit scl norm ball in $B_H^1(F)$ for a free group F is a rational polyhedron. We briefly mention the other best known example of polyhedral norm here and relate it to scl.

Theorem 2.14 (Thurston,[11])**.** *Let M be a compact oriented 3-manifold (For simplicity we assume ∂M is empty). Define the Thurston norm η on the integral lattice of $H_2(M)$ by*

$$\eta(a) = \inf\{-\chi^-(S) | S \textit{ is an embedded surface representing } a\}$$

Then,

1. *Using linearity and sub-additivity, the definition can be extended to define a pseudo-norm on $H_2(M;\mathbb{R})$ such that it is convex and linear on rays through the origin.*

2. *The unit ball B_{η^*} is a polyhedron with vertices at even lattice points $\pm\beta_1,\ldots,\beta_k$ and B_η is a polyhedron defined by linear inequalities with integer coefficients*

$$B_\eta = \{a \mid \ |a.\beta_i| \leq 1, 1 \leq i \leq k\}$$

We can relate the Thurston and Gromov norm by the proportionality theorem of Gabai which states that the Thurston norm η is exactly the half of Gromov norm $\|.\|_1$. This follows from the fact that a surface is Thurston norm minimizing iff it is a compact leaf of a taut foliation. Using the proportionality theorem we can deduce

Theorem 2.15 ([5])**.** *Let M be a compact oriented $3-$manifold. Let $\gamma \subset \partial M$ be an embedded, oriented loop. Let a be the conjugacy class in $\pi_1(M)$ represented by γ. Suppose $a \in [\pi_1(M), \pi_1(M)]$. Then $\mathrm{scl}(a) \in \mathbb{Q}$.*

Proof. (Sketch) Let N be obtained by doubling M along a regular annulus neighborhood A of γ. Let $V \subset H_2(N)$ be the integral affine subspace $V = \partial^{-1}([\gamma])$ where $[\gamma] \in H_1(A)$ is the generator. Then we can prove that

$$\mathrm{scl}(a) = \frac{1}{4}\mathrm{fill}(a) = \frac{1}{8}\inf_{v\in V}\|v\|_1 = \frac{1}{4}\inf_{v\in V}\eta(v)$$

□

Using the theory of branched surfaces, we can prove the *Rationality Theorem* which shows how scl has similar properties as Gromov-Thurston norm.

Theorem 2.16 (Calegari,[5])**.** *Let F be a free group. Then*

1. $\mathrm{scl}(g) \in \mathbb{Q}$ *for all* $g \in [F, F]$.
2. *Every $g \in [F, F]$ bounds an extremal surface.*
3. *The function* scl *is a piecewise linear norm on $B_1^H(F)$.*
4. *Every nonzero finite rational linear chain $A \in B_1^H(F)$ projectively bounds an extremal surface.*
5. *There is an algorithm to calculate* scl *on any finite dimensional rational subspace of $B_1^H(F)$, and to construct all extremal surfaces in a given projective class.*

The Rationality theorem enables us to relate scl to rotation and area quasimorphisms defined on arbitrary oriented surface with boundary, which we do in the next section.

§3. MORE EXAMPLES OF QUASIMORPHISMS

Quasimorphisms arise from hyperbolic geometry (negative curvature) and symplectic geometry (causal structures). We first look at the bounded area co-cycle appearing in hyperbolic geometry and the associated quasimorphisms. Then we discuss the famous rotation number quasimorphism and how it is related to scl. Finally we will consider quasimorphisms appearing in Symplectic geometry, namely the symplectic translation number.

◇ 3.1 de Rham quasimorphisms

Let M be a closed hyperbolic manifold, and let α be a 1-form. Define a quasimorphism $q_\alpha : \pi_1(M) \to \mathbb{R}$ as follows. Choose a base point $p \in M$. For each $\gamma \in \pi_1(M)$, let L_γ be the unique oriented geodesic arc with both endpoints at p which as a based loop represents γ in $\pi_1(M)$. Then define

$$q_\alpha(\gamma) = \int_{L_\gamma} \alpha$$

If γ_1, γ_2 are two elements of $\pi_1(M)$, there is a geodesic triangle T whose oriented boundary is the union of $L_{\gamma_1}, L_{\gamma_2}, L_{\gamma_2^{-1}\gamma_1^{-1}}$. Then by Stokes' theorem and Gauss-Bonnet theorem, we have

$$q_\alpha(\gamma_1) + q_\alpha(\gamma_2) - q_\alpha(\gamma_1\gamma_2) = \int_T d\alpha \Longrightarrow D(q_\alpha) \leq \pi.\|d\alpha\|$$

Note that the homogenization $\overline{q}_\alpha$ satisfies

$$\overline{q}_\alpha(\gamma) = \int_{l_\gamma} \alpha$$

where l_γ is the free geodesic loop corresponding to the conjugacy class of γ in $\pi_1(M)$. For, changing the base point p changes q_α by a bounded amount, and therefore does not change the homogenization.

In general the cohomology class of the volume cocycle of a hyperbolic $n-$manifold is bounded. We next describe the idea of Straightening chains (originally, due to Thurston) and formalize the construction of the quasimorphisms.

◇ 3.2 Straightening Chains

Definition 3.1. Let M be a hyperbolic $m-$manifold, $\sigma : \Delta^n \to M$ be a singular $n-$simplex. Define *straightening* σ_g of σ as follows. First lift σ to a map from $\Delta^n \to \mathbb{H}^m$, denote it by $\widetilde{\sigma}$.

Let $v_0, \ldots, v_n$ denote the vertices of Δ^n. Consider the hyperboloid model

$$\mathbb{H}^m = \{x_{m+1} > 0 : x_1^2 + \ldots x_m^2 - x_{m+1}^2 = -1\}$$

For $v = \sum_i t_i v_i \in \Delta^n$ (barycentric coordinates), define

$$\widetilde{\sigma}_g(v) = \frac{\sum t_i \widetilde{\sigma}(v_i)}{-\| \sum t_i \widetilde{\sigma}(v_i) \|}$$

and $\sigma_g = p \circ \widetilde{\sigma}_g$. Intuitively, it is the projection of the convex hull of $\{v_0, \ldots, v_n\}$.

Define

$$str : C_*(M) \to C_*(M)$$

by setting $str(\sigma) = \sigma_g$ and extending by linearity. By composing a linear homotopy in $\mathbb{R}^{m+1}$ with radial projection to the hyperboloid, we see that there is a chain homotopy between str and Id.

Lemma 3.2. $v_k = \sup Vol(\sigma)$ *over all geodesic simplices* $\sigma : \Delta^k \to \mathbb{H}^n$ *is finite for* $k \neq 1$.

Note 3.3. Haagerup-Munkholm proved that v_k is the volume of a regular ideal simplex.

Now consider a closed hyperbolic manifold M. We know that the natural inclusion induces a homomorphism from $H_b^n(M;\mathbb{R})$ to $H^n(M;\mathbb{R})$ for $n \geq 2$. Using straightening of chains and the fact that str and Id are homotopic we find that the homomorphisms is in fact a surjection i.e. every cohomology class of dimension n is in the image of a bounded cohomology class of dimension n. In the case $n = 2$, by the exact sequence 2.3.2.1 we find that an element of $H_b^2(M;\mathbb{R})$ whose image in $H^2(M;\mathbb{R})$ is trivial must be in the image of an element of $Q(\pi_1(M))$. Thus we can produce nontrivial quasimorphisms on M.

◇ 3.3 Rotation Number

Let Homeo(S^1) denote the group of homeomorphisms of the circle, and let Homeo$^+(S^1)$ be its orientation preserving subgroup. Let G be a subgroup of Homeo$^+(S^1)$. Let $\widetilde{G}$ be the preimage of G in Homeo$^+(\mathbb{R})$ under the covering projection $\mathbb{R} \to S^1$. Note that $\widetilde{G}$ is a central extension of G so that we have an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{G} \to G \to 1$$

Definition 3.4 (Poincaré)**.** Given $g \in \widetilde{G}$, define the *rotation number* of g to be

$$\mathrm{rot}(g) = \lim_{n\to\infty} \frac{g^n(0)}{n}$$

The rotation number of an element $\varphi \in G$ is defined to be $\mathrm{rot}(\widetilde{\varphi}) \mod \mathbb{Z}$, where $\widetilde{\varphi}$ is a lift of φ in $\widetilde{G}$.

Lemma 3.5. rot *is a quasimorphism on* $\widetilde{G}$.

Proof. Given arbitrary a, b, write $a = \mathbb{Z}^n a', b = \mathbb{Z}^m b'$ where $0 \leq a'(0), b'(0) < 1$. Then $ab = \mathbb{Z}^{m+n} a'b'$ and hence

$$\mathrm{rot}(a) + \mathrm{rot}(b) - \mathrm{rot}(ab) = n + \mathrm{rot}(a') + m + \mathrm{rot}(b') - m - n - \mathrm{rot}(a'b') \Rightarrow D(\mathrm{rot}) \leq 2$$

□

In fact we can prove more precisely that

Lemma 3.6. *For all $p \in \mathbb{R}$ and $a, b \in \widetilde{G}$, there is an inequality*

$$p - 2 < [a, b]p < p + 2$$

Thus it follows that $\mathrm{scl}(a) \geq \frac{|\mathrm{rot}(a)|}{2}$ for any $a \in \widetilde{G}$. In fact we have an equality.

Theorem 3.7. *Let $\widetilde{\mathrm{Homeo}}^+(S^1)$ denote the set consisting of all possible lifts of elements in $\mathrm{Homeo}^+(S^1)$ under the covering projection $\mathbb{R} \to S^1$. Then*

$$\mathrm{scl}(a) = \frac{|\mathrm{rot}(a)|}{2} \text{ for all } a \in \widetilde{\mathrm{Homeo}}^+(S^1).$$

Before proving the theorem, we need to introduce the following definition.

Definition 3.8. A group is called *uniformly perfect* if every element can be written as a product of bounded number of commutators.

Observe that any quasimorphism on a uniformly perfect group is bounded which easily follows from estimate 2.3.1.1. Consequently, for any uniformly perfect group Γ, the canonical maps $H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ and $H_b^2(\Gamma, \mathbb{Z}) \to H_b^2(\Gamma, \mathbb{R})$ are injective. Thus for uniformly perfect groups, the usual Euler class in $H^2(\Gamma, \mathbb{Z})$ determines the bounded Euler class. By a theorem proved by Eisenbud, Hirsch and Neumann in [8], we can prove that $\mathrm{Homeo}^+(S^1)$ is uniformly perfect; every element can be written as a commutator. Then the proof of theorem 3.7 follows from the following two lemmas.

Lemma 3.9. $\mathrm{rot} : \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$ *is the unique homogeneous quasimorphism which sends the unit translation to* 1.

Proof. Suppose $\tau \in Q(\widetilde{\mathrm{Homeo}}^+(S^1))$ is another such map, then we consider

$$r = \mathrm{rot} - \tau : \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$$

which is also a homogeneous quasimorphism, and since any homogeneous quasimorphism on abelian groups, specifically $\mathbb{Z} \oplus \mathbb{Z}$, must be a homomorphism, we have $r(f_1) = r(f_2)$, if f_1 and f_2 descend to the same function in $\mathrm{Homeo}^+(S^1)$. Thus it induces a homogeneous quasimorphism $\overline{r}$ on $\mathrm{Homeo}^+(S^1)$. But $\overline{r}$ is bounded on $\mathrm{Homeo}^+(S^1)$ by above discussion. Since it's homogeneous, it must be the zero map, i.e. $\mathrm{rot} = \tau$. $\square$

It can be proved using Bavard's lemma 2.11 that

Lemma 3.10 ([12])**.** $D(\mathrm{rot}) = 1$.

Proof of Theorem 3.7. Observe that for $G = \widetilde{\mathrm{Homeo}}^+(S^1)$, we have $\dim_{\mathbb{R}}(Q(G)) = 1$. Then lemma 3.9 and lemma 3.10 together, along with Bavard's duality theorem 2.13, imply theorem 3.7. $\square$

We can similarly define a rotation number on the group $PSL(2, \mathbb{R}) \subseteq \mathrm{Homeo}^+(S^1)$, when it is considered to be acting on the circle at infinity of hyperbolic space. In fact, for a compact oriented hyperbolic surface S and a discrete faithful representation $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$ we can pull back the rotation number by $\widetilde{\rho}$ to define a function rot_S on $\pi_1(S)$. Here $\widetilde{\rho}$ is a lift of ρ to $\widetilde{SL}(2, \mathbb{R}) \subset Homeo^+(\mathbb{R})$. Since different lifts are classified by elements of $H^1(S; \mathbb{Z})$, rot_S is well defined on the commutator subgroup of $\pi_1(S)$ independent of choice of $\widetilde{\rho}$. The following lemma says that the algebraic area of a geodesic is actually a rotation number.

Lemma 3.11. *If $g \in [\pi_1(S), \pi_1(S)]$ be represented by a geodesic $\gamma \subset S$. Then*

$$\mathrm{area}(\gamma) = -2\pi.\mathrm{rot}_S(g)$$

Proof. (Sketch) Let $f ; (S', \partial S') \to (S, \gamma)$ be a pleated surface with $n(S') = 1$. Then

$$\frac{\mathrm{area}(\gamma)}{2\pi} = e\left(f_*[S']\right) = -\left(\delta\, \mathrm{rot}_S\right)\left(f_*[S']\right) = -\mathrm{rot}_S\left(f_*[\partial S']\right) = -\mathrm{rot}_S(g)$$

where $[e] = -[\delta\, \mathrm{rot}_S] \in H_b^2(\pi_1(S); \mathbb{R})$ is the Euler class. $\square$

In general if a chain $C = \sum t_i a_i$ in $B_1^H(F)$ is represented by a 'weighted' union Γ of geodesics then $\text{area}(\Gamma) = -2\pi \sum t_i \text{rot}_S(a_i)$. Then using Bavard Duality theorem 2.13 and Rationality theorem 2.16 we can relate scl and rot_S as follows:

Theorem 3.12 ([5])**.** *Let S be a oriented hyperbolic surface with boundary. Let C be a rational chain in $B_1^H(S)$ represented by a weighted sum of geodesics Γ. Then Γ rationally bounds a (positive or negative) immersed subsurface S if and only if*

$$\text{scl}(C) = \frac{|\text{rot}_S(C)|}{2}$$

i.e. rot_S is an extremal quasimorphism for C.

In the next section, we try to generalize this notion of rotation quasimorphisms to Symplectic groups.

$$\begin{tikzcd}
 & \widetilde{Sp}(2n,\mathbb{R}) \arrow[dr, dash] & & \widetilde{Homeo}^+(S^1) \arrow[dl, dash] & Sp(2n,\mathbb{R}) \arrow[dr, dash] & & Homeo^+(S^1) \arrow[dl, dash] & \\
0 \longrightarrow \mathbb{Z} \longrightarrow & & & \longrightarrow & & & & \longrightarrow 1 \\
 & & \widetilde{PSL}(2,\mathbb{R}) \arrow[d, squiggly, "\text{rot}"] & & & PSL(2,\mathbb{R}) \arrow[d, squiggly, "\text{rot}"] & & \\
 & & \mathbb{R} & & & \mathbb{R}/\mathbb{Z} & &
\end{tikzcd}$$

◇ 3.4 Quasimorphism arising from Symplectic Geometry

We equip $\mathbb{R}^{2n}$ with the coordinates $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ and consider the symplectic form ω on $\mathbb{R}^{2n}$ defined by

$$\omega\left((q_\alpha, p_\alpha), (q'_\alpha, p'_\alpha)\right) = \sum_{\alpha=1}^{n} \det \begin{bmatrix} q_\alpha & q'_\alpha \\ p_\alpha & p'_\alpha \end{bmatrix}$$

The Symplectic group $Sp(2n, \mathbb{R})$ is the group of linear automorphisms of $\mathbb{R}^{2n}$ which preserve ω. The quotient of $Sp(2n, \mathbb{R})$ by the center $\{\pm Id\}$ is denoted by $PSp(2n, \mathbb{R})$.

3.4.1 | Quasimorphisms on $\widetilde{Sp}(2n, \mathbb{R})$ and Maslov Class

Proposition 3.13. *$Sp(2n, \mathbb{R})$ is uniformly perfect.*

Proof. For $a \in \mathbb{R}^{2n}$, we consider the linear map τ_a defined for any $x \in \mathbb{R}^{2n}$ by

$$\tau_a(x) = x + \omega(x, a)a$$

The maps τ_a are called *symplectic transvections*. Let $g \in Sp(2n, \mathbb{R})$ be such that $g(a) = \sqrt{2}.a$. Then

$$\tau_a = (\tau_a)^2 \tau_a^{-1} = \tau_{\sqrt{2}.a}.\tau_a^{-1} = g\tau_a g^{-1}\tau_a^{-1}$$

Thus τ_a is a commutator. The proposition then follows from the fact that any element of $Sp(2n, \mathbb{R})$ is the product of at most $2n$ elements of type $\tau_a^{\pm 1}$([7]). □

Definition 3.14. A $n-$dimensional subspace L of $\mathbb{R}^{2n}$ is called a *Lagrangian* if the restriction of ω to $L \times L$ is identically zero. We denote the space of all Lagrangian subspaces of $\mathbb{R}^{2n}$, called the Lagrangian grassmannian, by Λ_n.

Clearly, $PSp(2n, \mathbb{R})$ acts on Λ_n. One can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the coordinates $z_\alpha = q_\alpha + ip_\alpha(\alpha = 1, \ldots, n)$. Then the Unitary group $U(n)$ which preserves the standard hermitian form $\sum z_\alpha \bar{z}_\alpha$ can be though of as a subspace of $Sp(2n, \mathbb{R})$. It turns out that $U(n)$ acts transitively on Λ_n and the stabilizer of this action is the orthogonal group $O(n)$. Thus we can identify

$$\Lambda_n \cong U(n)/O(n)$$

The square of the determinant map $d = \det^2$ defines a fibration

$$d = \det{}^2 : \Lambda_n = U(n)/O(n) \to S^1 \subset \mathbb{C}^*, \quad L = A(\mathbb{R}^n) \mapsto \det(A)^2$$

Each fiber is diffeomorphic to $SU(n)/SO(n)$ and is simply connected.

Proposition 3.15. *The fibration induces an isomorphism at the level of fundamental groups. Thus the fundamental group of the Lagrangian grassmannian is infinite cyclic.*

Proof. (Sketch) The proof follows from the homotopy exact sequence of the commutative diagram of fiber bundles

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & O(n) \arrow[r, "\det"] \arrow[d] & O(1) = S^0 \arrow[d] \\ SU(n) \arrow[r] \arrow[d] & U(n) \arrow[r, "\det"] \arrow[d] & U(1) = S^1 \arrow[d, "z \mapsto z^2"] \\ S\Lambda_n \arrow[r] & \Lambda_n \arrow[r, "d=\det^2"] & \Lambda_1 = S^1 \end{tikzcd}$$

Hence

$$0 \longrightarrow \pi_1(U(n)/O(n)) \xrightarrow[\cong]{d_*} \pi_1(S^1) \longrightarrow 0$$

□

Corollary 3.16. $H_1(\Lambda_n)$ *is infinite cyclic.*

Let $\pi : \widetilde{\Lambda}_n \to \Lambda_n$ be the universal cover of Λ_n. Let $\widetilde{Sp}(2n, \mathbb{R})$ be the subgroup of homeomorphisms of $\widetilde{\Lambda}_n$ which descend to action of $PSp(2n, \mathbb{R})$ on Λ_n. We have an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{Sp}(2n, \mathbb{R}) \to PSp(2n, \mathbb{R}) \to 1$$

It turns out that $\widetilde{Sp}(2n, \mathbb{R})$ is the universal cover of $PSp(2n, \mathbb{R})$. The cohomology class in $H^2(PSp(2n, \mathbb{R}); \mathbb{Z})$ that defines this extension is defined to be the *Maslov class.*

In the case $n = 1$, when we have

$$0 \to \mathbb{Z} \to \widetilde{SL}(2, \mathbb{R}) \to PSL(2, \mathbb{R}) \to 1$$

the Maslov class turns out to be the same as the Euler class and Λ_1 becomes a circle. The quasimorphisms on $\widetilde{SL}(2, \mathbb{R})$ come mainly from the choice of fundamental domain for the action of integer translations. In the remaining part of this section, we talk about ways to generalize the quasimorphisms.

First Method: Let L be the lagrangian $\mathbb{R}^n \times \{0\}$ and $\widetilde{L}$ be its lift in $\widetilde{\Lambda}_n$. Let $\widetilde{d} = \widetilde{\det}^2 : \widetilde{\Lambda}_n \to \mathbb{R}$ be a lift of d such that $\widetilde{d}(\widetilde{L}) = 0$. We define the function $\Phi_{\det}$ on $\widetilde{Sp}(2n, \mathbb{R})$ by

$$\Phi_{\det}(\widetilde{g}) = \widetilde{\det}^2(\widetilde{g}(\widetilde{L})) \in \mathbb{R}$$

We can prove that,

Proposition 3.17 ([2])**.** $\Phi_{\det}$ *is an quasimorphism.*

The bounded 2−cocycle $c_{\det}$ defined on $PSp(2n, \mathbb{R})$ by

$$c_{\det}(g_1, g_2) = \Phi_{\det}(\widetilde{g}_1\widetilde{g}_2) - \Phi_{\det}(\widetilde{g}_1) - \Phi_{\det}(\widetilde{g}_2)$$

represents the Maslov class. By replacing $\Phi_{\det}$ by its integer part, we obtain an integer 2−cocycle also representing the Maslov class and having values less that $n + 1$.

Second Method: Another method of determining an 2−cocycle representing the Maslov class was given by Arnold[1]. Let $\alpha \in \Lambda_n$ be a Lagrangian.

Definition 3.18. The *train* of α, denoted $t(\alpha)$ is the set of all Lagrangian planes which are not transverse to α. α is called the vertex of the train $t(\alpha)$.

Every train is a codimension 1 subvariety in Λ_n, whose singularities form a set of codimension 2 in the train. In a neighborhood of the point α of $t(\alpha)$, the elements of Λ_n correspond to quadratic forms with the elements of $t(\alpha)$ corresponding to degenerate ones. Consequently, $t(\alpha)$ divides the neighborhood of the vertex into $(2n+1)$ disjoint subsets corresponding to the signature of the quadratic form. In particular, in the neighborhood of the vertex, there is a distinguished "positive" domain whose corresponding quadratic forms are positive definite. The field of such domains as α varies defines a causal structure on Λ_n.

Definition 3.19. If $\gamma : [0,1] \to \Lambda_n$ is a 1−parameter family of Lagrangian subspaces, with $\gamma(0) = \alpha$ and $\gamma(1) = \beta$, then the Maslov index of γ is the algebraic intersection number of a path starting at α and ending at a point β' lying in the 'positive' domain near β, with the train of α.

The homotopy class of a path joining α and β can be represented by a pair of points $\widetilde{\alpha}$ and $\widetilde{\beta}$ in $\widetilde{\Lambda}_n$. We denote the Maslov index of such a path by $m(\widetilde{\alpha}, \widetilde{\beta})$. If T denotes the generator of the action of the deck group $\mathbb{Z}$ on $\widetilde{\Lambda}_n$, then we have the formula:

$$m(T\widetilde{\alpha}, \widetilde{\beta}) = m(\widetilde{\alpha}, \widetilde{\beta}) + 1$$

This is clear because the path corresponding to $(T\widetilde{\alpha}, \widetilde{\beta})$ is obtained from the path corresponding to $(\widetilde{\alpha}, \widetilde{\beta})$ by adding a loop whose intersection number with the train of the plane $\pi(\widetilde{\beta})$ equals 1.

Definition 3.20. For each pair α, β of transverse Lagrangian plane in $(\mathbb{R}^{2n}, \omega)$, we consider the adjoined quadratic form $\Phi[\alpha, \beta]$ in $\mathbb{R}^{2n}$, whose value on any vector ζ is defined by

$$\Phi[\alpha, \beta](\zeta) = \omega(z_1, z_2) \text{ where } \zeta = z_1 + z_2, z_1 \in \alpha, z_2 \in \beta$$

It is easy to see that Φ is symmetric. Φ and the Maslov index are related via the following relation.

Definition 3.21. The index $I(\alpha, \beta, \gamma)$ of the triplet of Lagrangians α, β, γ is the signature of the restriction of the form $\Phi[\alpha, \beta]$ to γ.

Then we can write,

$$I(\pi(\widetilde{\alpha}), \pi(\widetilde{\beta}), \pi(\widetilde{\gamma})) = m(\widetilde{\alpha}, \widetilde{\beta}) + m(\widetilde{\beta}, \widetilde{\gamma}) + m(\widetilde{\gamma}, \widetilde{\alpha}) - n$$

for any $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ in $\widetilde{\Lambda}_n$. We define the Arnold-Maslov function

$$\Phi_{AM} : \widetilde{g} \in \widetilde{Sp}(2n, \mathbb{R}) \mapsto m(\widetilde{g}(\widetilde{L}), \widetilde{L})$$

where $L = \mathbb{R}^n \times \{0\}$ as defined in last section. The associated cocycle c_{AM} for $PSp(2n, \mathbb{R})$ defined by

$$c_{AM}(g_1, g_2) = \Phi_{AM}(\widetilde{g}_1\widetilde{g}_2) - \Phi_{AM}(\widetilde{g}_1) - \Phi_{AM}(\widetilde{g}_2) = I(L, g_1(L), g_1g_2(L)) - n$$

is then the Maslov class.

Proposition 3.22. *Φ_{AM} is a quasimorphism.*

Proof. This follows from the fact that the index I is bounded by $2n$. □

Corollary 3.23. *The defects of $\Phi_{\det}$ and Φ_{AM} are different.*

3.4.2 | Symplectic Rotation Number

Definition 3.24. The symplectic rotation number of an element $\widetilde{g}$ of $\widetilde{Sp}(2n, \mathbb{R})$ is defined by

$$\operatorname{rot}(\widetilde{g}) = \lim_{n\to\infty} \frac{\Phi_{\det}(\widetilde{g}^n)}{n} = \lim_{n\to\infty} \frac{\Phi_{AM}(\widetilde{g}^n)}{n}$$

and it descends to a symplectic rotation number of an element $\pi(\widetilde{g}) = g \in PSp(2n, \mathbb{R})$ by

$$\operatorname{rot}(g) = \operatorname{rot}(\widetilde{g}) \mod \mathbb{Z}$$

§4. REFERENCES

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