*Ergod. Th. & Dynam. Sys.* (2017), **37**, 739–757 
doi:10.1017/etds.2015.75

# Ziggurat fringes are self-similar

SUBHADIP CHOWDHURY

*Department of Mathematics, University of Chicago, Chicago, IL 60637, USA*
*(e-mail: subhadip@math.uchicago.edu)*

(*Received* 1 *April* 2015 *and accepted in revised form* 27 *July* 2015)

*Abstract*. We give explicit formulae for fringe lengths of the Calegari–Walker ziggurats—i.e., graphs of extremal rotation numbers associated with positive words in free groups. These formulae reveal (partial) integral projective self-similarity in ziggurat fringes, which are low-dimensional projections of characteristic polyhedra on the bounded cohomology of free groups. This explains phenomena observed experimentally by Gordenko, Calegari and Walker.

## 1. *Introduction*

Let $\text{Homeo}^{\sim}_{+}(S^1)$ denote the group of homeomorphisms of the real line that commute with integer translation, and let $\text{rot}^{\sim} : \text{Homeo}^{\sim}_{+}(S^1) \to \mathbb{R}$ denote Poincaré's (real-valued) rotation number, also known as translation number. Let $F$ be a free group on two generators $a$, $b$ and let $w$ be a word in the semigroup generated by $a$ and $b$ (such a $w \in F$ is said to be *positive*). Let $h_a(w)$ and $h_b(w)$ be the number of $a$ and $b$ in $w$, respectively. The *fringe* associated with $w$ and a rational number $0 \le p/q < 1$ is the set of $0 \le t < 1$ for which there is a homomorphism from $F$ to $\text{Homeo}^{\sim}_{+}(S^1)$ with $\text{rot}^{\sim}(a) = p/q$, $\text{rot}^{\sim}(b) = t$ and $\text{rot}^{\sim}(w) = h_a(w)p/q + h_b(w)$. Calegari and Walker [**4**] show that there is some least rational number $s \in [0, 1)$ so that the fringe associated with $w$ and with $p/q$ is equal to the interval $[s, 1)$. The *fringe length*, denoted $\text{fr}_w(p/q)$, is equal to $1 - s$.

The main theorem that we prove in this paper is an explicit formula for fringe length.

*Fringe Formula 1.3. If $w$ is positive, and $p/q$ is a reduced fraction, then*

$$\text{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q}$$

*where $\sigma_w(g)$ depends on the word $w$ and on $g := gcd(q, h_a(w))$. Furthermore, $g \cdot \sigma_w(g)$ is an integer.*

As $t \to 1$, the dynamics of $F$ on $S^1$ are approximated more and more closely by a linear model, as explained below by Theorem 1.2. For $t$ close to 1, the nonlinearity can be characterized by a perturbative model; fringes are the maximal regions where this perturbative model is valid. Our main theorem says that the size of this region of stability follows a power law. This is a new example of (topological) nonlinear phase locking in 1-dimensional dynamics giving rise to a power law, of which the most famous example is the phenomenon of Arnol'd Tongues [**5**].

1.1. *Motivation.* If $G$ is a Lie group, and $\Gamma$ is a finitely generated group, one studies representations of $\Gamma$ into $G$ up to conjugacy, not by looking at the quotient space $\mathrm{Hom}(\Gamma, G)/G$ (which is usually non-Hausdorff), but by taking a further (maximal) quotient on which certain natural functions—*characters*—are continuous and well-defined (i.e., one studies *character varieties*).

Recovering a representation from a character is not always straightforward. Given a (finite) subset $S$ of $\Gamma$, it becomes an interesting and subtle question to ask what constraints are satisfied by the values of a character on $S$. For example, the (multiplicative) Horn problem poses the problem of determining the possible values of the spectrum of the product AB of two unitary matrices given the spectra of A and B individually. There is a map

$$\Lambda : SU(n) \times SU(n) \to \mathbb{R}^{3n}$$

taking $A$, $B$ to the logarithms of the spectra of $A$, $B$ and $AB$ (suitably normalized). Agnihotri and Woodward [**1**] and Belkale [**2**] proved that the image is a convex polytope, and explicitly described the image.

When $G$ is replaced by a topological group such as $\mathrm{Homeo}^{\sim}_{+}(S^1)$ (the group of orientation-preserving homeomorphisms of the circle), the situation becomes more complicated. Recall that the translation number

$$\mathrm{rot}^{\sim} : \mathrm{Homeo}^{\sim}_{+}(S^1) \to \mathbb{R}$$

is constant on conjugacy classes (more precisely, on semi-conjugacy classes: see, for example, [**6**] or [**3**] and §2.1 for more details) and can be thought of as the analog of a character in this context. Following Calegari and Walker [**4**] we would like to understand what constraints are simultaneously satisfied by the value of $\mathrm{rot}^{\sim}$ on the image of a finite subset of $\Gamma$ under a homomorphism to $\mathrm{Homeo}^{\sim}_{+}(S^1)$. That is, we study the values $x_i := \mathrm{rot}^{\sim}(\rho(w_i))$ for a finite number of $w_i \in \Gamma$ on a common representation $\rho$.

1.2. *Free groups, positive words and ziggurats.* The universal case to understand is that of a free group. For simplicity, we consider the case of a rank 2 free group. Thus, let $F$ be a free group with generators $a$, $b$, and for any element $w \in F$ let $x_w$ be the function from conjugacy classes of representations $\rho : F \to \mathrm{Homeo}^{\sim}_{+}(S^1)$ to $\mathbb{R}$ which sends a representation $\rho$ to $x_w(\rho) := \mathrm{rot}^{\sim}(\rho(w))$. The $x_w$ are *coordinates* on the space of conjugacy classes of representations, and we study this space through its projections to finite dimensional spaces obtained from a finite number of these coordinates.

For any $w \in F$ and for any $r, s \in \mathbb{R}$ we can define

$$X(w; r, s) = \{x_w(\rho) \mid x_a(\rho) = r, x_b(\rho) = s\}.$$

Then $X(w; r, s)$ is a *compact* interval (see [**4**]) (i.e., the extrema are achieved) and it satisfies $X(w; r + m, s + n) = X(w; r, s) + mh_a(w) + nh_b(w)$, where $h_a, h_b : F \to \mathbb{Z}$ count the signed number of copies of $a$ and $b$ respectively in each word.

If we define $R(w; r, s) = \max\{X(w; r, s)\}$, then $\min\{X(w; r, s)\} = -R(w; -r, -s)$. So all the information about $X(w; r, s)$ can be recovered from the function $R(w; \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$. In fact, by the observations made above, it suffices to restrict the domain of $R$ to the unit square $[0, 1) \times [0, 1)$.

The theory developed in [**4**] is most useful when $w$ is a *positive word* (i.e., a word in the *semigroup* generated by $a$ and $b$). In this case, $R(w; r, s)$ is lower semi-continuous, and monotone non-decreasing in both its arguments. Furthermore, it is *locally constant* and takes *rational values* on an open and dense subset of $\mathbb{R}^2$. In fact, we have the following.

THEOREM 1.1. [**4**, Theorems 3.4, 3.7] *Suppose $w$ is positive (and not a power of $a$ or $b$), and suppose $r$ and $s$ are rational. Then:*

(1) *$R(w; r, s)$ is rational with denominator no bigger than the smallest of the denominators of $r$ and $s$; and*
(2) *there is some $\epsilon(r, s) > 0$ so that $R(w; \cdot, \cdot)$ is constant on $[r, r + \epsilon) \times [s, s \times \epsilon)$.*

Furthermore, when $r$ and $s$ are rational and $w$ is positive, Calegari and Walker [**4**] give an explicit combinatorial algorithm to compute $R(w; r, s)$; it is the existence and properties of this algorithm that proves Theorem 1.1. Computer implementation of this algorithm allows one to draw pictures of the graph of $R$ (restricted to $[0, 1) \times [0, 1)$) for certain short words $w$, producing a stairstep structure dubbed a *Ziggurat* (see Figure 1).

In the special case of the word $w = ab$, a complete analysis can be made, and an explicit formula obtained for $R(ab; \cdot, \cdot)$ (this case arose earlier in the context of the classification of taut foliations of Seifert fibered spaces, where the formula was conjectured by Jankins and Neumann [**8**] and proved by Naimi [**10**]). But in *no other case* is any explicit formula known or even conjectured, and even the computation of $R(w; r, s)$ takes time which is an exponential function of the denominators of $r$ and $s$.

1.3. *Projective self-similarity and fringes.* In a recent preprint, Gordenko [**7**] gave a new analysis and interpretation of the $ab$ formula, relating it to the Naimi formula in an unexpected way. Her formulation exhibits and explains an *integral projective self-similarity* of the $ab$-ziggurat, related to the theory of continued fractions, and the fact that the automorphism group of $F_2$ is $\mathrm{SL}(2, \mathbb{Z})$. Such global self-similarity is (unfortunately) not evident in ziggurats associated with other positive words; but there is a partial self-similarity (observed experimentally by Calegari and Walker [**4**] and by Gordenko [**7**]) in the *germ* of the ziggurats near the *fringes* where one of the coordinates $r$ or $s$ approaches 1 from below.

If we fix a positive word $w$ and a rational number $r$, and (following [**4**]) we denote by $R(w; r, 1-)$ the limit of $R(w; r, t)$ as $t \to 1$ from below, then the following can be proved.

THEOREM 1.2. [**4**, Proposition 3.15] *If $w$ is positive and $r$ is rational, there is a least rational number $s \in [0, 1)$ so that $R(w; r, t)$ is constant on the interval $[s, 1)$ and equal to $R(w; r, 1-) = h_a(w)r + h_b(w)$.*

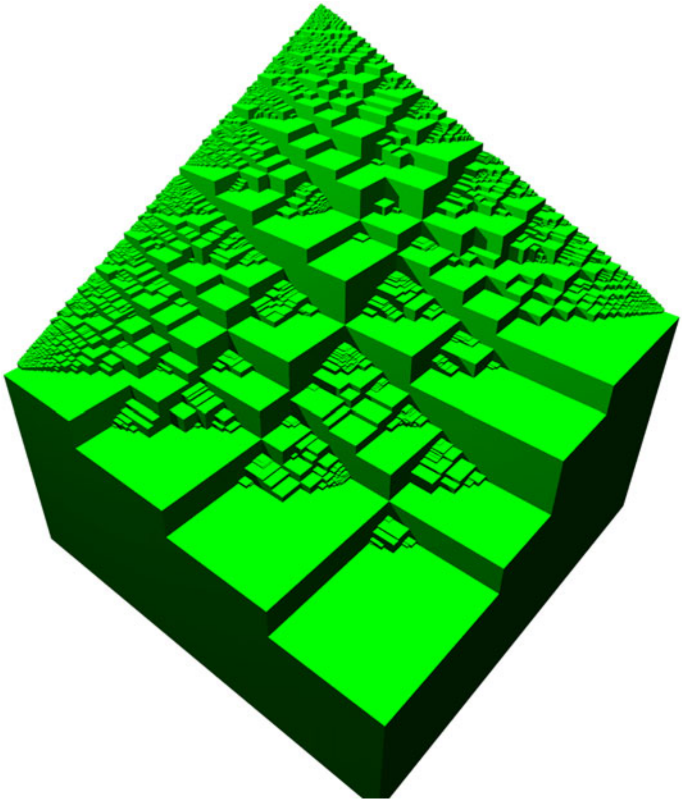

FIGURE 1. Graph of $R(abbbabaaaabbabb; \cdot, \cdot)$: colloquially, a *ziggurat*. Picture courtesy of Calegari and Walker [**4**].

We refer to the rational number $1 - s$ as in Theorem 1.2 (depending on the word $w$ and the rational number $r$) as the *fringe length* of $r$, and denote it by $\mathrm{fr}_w(r)$, or just by $\mathrm{fr}(r)$ if $w$ is understood. In other words, $\mathrm{fr}_w(r)$ is the greatest number such that $R(w; r, 1 - \mathrm{fr}_w(r)) = h_a(w)r + h_b(w)$. More precisely, we should call this a 'left fringe', where the right fringe should be the analog with the roles of the generators $a$ and $b$ interchanged.

1.4. *Statement of results.* Section 2 summarizes background, including some elements from the theory of ziggurats from [**4**]. The most important ingredient is a description of the stairstep algorithm.

In §3 we undertake an analysis of the stairstep algorithm when applied to the computation of fringe lengths. A number of remarkable simplifications emerge which allows us to reduce the analysis to a tractable combinatorial problem which depends (in a complicated way) only on $\gcd(q, h_a(w))$.

Our main theorem gives an explicit formula for $\mathrm{fr}_w$ for any positive word $w$, and establishes a (partial) integral projective self-similarity for a fringe. Thus, it gives a theoretical basis for the experimental observations of Calegari and Walker [**4**] and Gordenko [**7**].

THEOREM 1.3. (Fringe formula) *If $w$ is positive and $p/q$ is a reduced fraction, then*

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q},$$

*where $\sigma_w(g)$ depends only on the word $w$ and $g := \gcd(q, h_a(w))$; and $g \cdot \sigma(q)$ is an integer.*

The function $\sigma_w(g)$ depends on $w$ and on $q$ in a complicated way, but there are some special cases which are easy to understand. In §4 we prove the following inequality.

THEOREM 1.4. ($\sigma$-inequality) *Suppose $w = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_n} b^{\beta_n}$. Then the function $\sigma_w(g)$ satisfies the inequality*

$$\frac{h_b(w)}{h_a(w)} \leq \sigma_w(g) \leq \max \beta_i.$$

*Moreover, $h_b(w)/h_a(w) = \sigma_w(g)$ when $h_a(w)$ divides $q$, and $\sigma_w(g) = \max \beta_i$ when $q$ and $h_a(w)$ are coprime.*

The fringe formula explains the fact that $\mathrm{fr}_w(p/q)$ is independent of $p$ (for $\gcd(p, q) = 1$) and implies a periodicity of $\mathrm{fr}_w$ on an infinite number of scales. More precise statements are found in §5.

## 2. *Background*

### 2.1. *Rotation numbers.*

Consider the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}_+^\sim(S^1) \to \mathrm{Homeo}_+(S^1) \to 0,$$

which has center generated by unit translation $z : p \to p + 1$.

Poincaré defined the *rotation number* $\mathrm{rot} : \mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$ as

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n} \quad (\mathrm{mod}\ \mathbb{Z}),$$

where $\tilde{f}$ is a lift of $f$ in $\mathrm{Homeo}_+^\sim(S^1)$. More generally, if $g \in \mathrm{Homeo}_+^\sim(S^1)$ is a homeomorphism of the real line commuting with integer translations, its *translation number* $\mathrm{rot}^\sim : \mathrm{Homeo}_+^\sim(S^1) \to \mathbb{R}$ is defined as

$$\mathrm{rot}^\sim(g) = \lim_{n\to\infty} \frac{g^n(0)}{n}.$$

Since $\mathrm{rot}^\sim(gz^n) = \mathrm{rot}^\sim(g) + n$ for any integer $n$, we get that $\mathrm{rot}^\sim$ descends to a well-defined function $\mathrm{rot} : \mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$.

Recall that, for $F$ a free group generated by $a$, $b$, for any $w \in F$ and for any numbers $r, s \in \mathbb{R}$, we define $R(w; r, s)$ to be the maximum value of $\mathrm{rot}^\sim(\rho(w))$ under all homomorphisms $\rho : F \to \mathrm{Homeo}_+^\sim(S^1)$ for which $\mathrm{rot}^\sim(\rho(a)) = r$ and $\mathrm{rot}^\sim(\rho(b)) = s$. The maximum is achieved on some representation $\rho$ for any fixed $r$ and $s$ [**4**, Lemma 2.13], but the function $R(w; \cdot, \cdot)$ is typically not continuous in either $r$ or $s$.

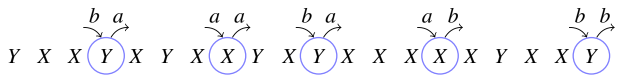


FIGURE 2. Action of $a$ and $b$ on $W$.

2.2. *Positive words and $XY$ words.* Now suppose $w$ is a positive word (i.e., containing only positive powers of $a$ and $b$), and $r = p_1/q_1$, $s = p_2/q_2$ are rational and expressed in reduced form. Theorem 1.1 says that $R(w;\, p_1/q_1,\, p_2/q_2)$ is rational, with denominator no bigger than $\min(q_1, q_2)$. Following [**4**], we present the Calegari–Walker algorithm to compute $R(w;\, p_1/q_1,\, p_2/q_2)$. The main idea is that since the rotation number essentially encodes the cyclic combinatorial order of the orbits in the circle, we can find $R(w;\, p_1/q_1,\, p_2/q_2)$ using purely combinatorial methods.

*Definition 2.1.* ($XY$-word) An $XY$-word of *type* $(q_1, q_2)$ is a cyclic word in the 2-letter alphabet $X$, $Y$ of length $q_1 + q_2$, with a total of $q_1$ $X$s and $q_2$ $Y$s.

If $W$ is an $XY$-word of type $(q_1, q_2)$, we let $W^\infty$ denote the bi-infinite string obtained by concatenating $W$ an infinite number of times, and think of this bi-infinite word as a function from $\mathbb{Z}$ to $\{X, Y\}$: we denote the image of $i \in \mathbb{Z}$ under this function by $W_i$, so that each $W_i$ is an $X$ or a $Y$, and $W_{i+q_1+q_2} = W_i$ for any $i$.

We define an action of the *semigroup* generated by $a$ and $b$ on $\mathbb{Z}$, associated to the word $W$ (see Figure 2). The action is given as follows. For each integer $i$, we define $a(i) = j$ where $j$ is the least index such that the sequence $W_i, W_{i+1}, \ldots, W_j$ contains exactly $p_1 + 1$ $X$s. Similarly, $b(i) = j$ where $j$ is the least index such that the sequence $W_i, W_{i+1}, \ldots, W_j$ contains exactly $p_2 + 1$ $Y$s. Note that this means $W_{a(i)}$ is always an $X$ and $W_{b(i)}$ is always $Y$, respectively. We can then define

$$\mathrm{rot}^{\sim}_{W}(w) = \lim_{n\to\infty} \frac{w^n(1)}{n \cdot (q_1 + q_2)}.$$

PROPOSITION 2.2. (Calegari–Walker formula) *With notation as above, there is a formula*

$$R(w;\, p_1/q_1,\, p_2/q_2) = \max_W \{\mathrm{rot}^{\sim}_{W}(w)\},$$

*where the maximum is taken over the* finite *set of $XY$-words $W$ of type* $(q_1, q_2)$.

Evidently, each $\mathrm{rot}^{\sim}_{W}(w)$ is rational, with denominator less than or equal to $\min(q_1, q_2)$, proving the first part of Theorem 1.1. Although theoretically interesting, a serious practical drawback of this proposition is that the number of $XY$-words of type $(q_1, q_2)$ grows exponentially in the $q_i$.

2.3. *Stairstep algorithm.* In this subsection we discuss the stairstep algorithm, found in [**4**], in more detail and in the context of this paper.

THEOREM 2.3. [**4**, Theorem 3.11] *Let $w$ be a positive word, and suppose $p/q$ and $c/d$ are rational numbers so that $c/d$ is a value of $R(w;\, p/q, \cdot)$. Then*

$$u := \inf\{t : R(w;\, p/q, t) = c/d\}$$

*is rational, and $R(w;\, p/q, u) = c/d$.*

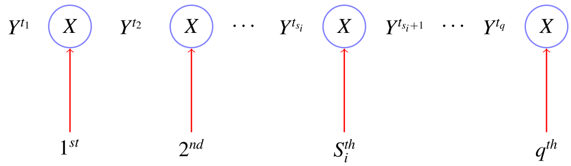


FIGURE 3. The $XY$ word of type $(q, v)$.

The theorem is proved by giving an algorithm (the stairstep algorithm) to compute $u$ and analyzing its properties. Note that the fringe length $\mathrm{fr}_w(p/q)$ is the value of $1-u$ where $u$ is the output of the stairstep algorithm for $c/d = h_a(w)p/q + h_b(w)$. Observe that, whereas Theorem 1.2 proved the existence of a fringe length, this theorem proves that the length is in fact a rational number. We now explain this algorithm.

*Proof.* Since $R$ is monotone non-decreasing in both of its arguments, it suffices to prove that

$$\inf\{t : R(w;\ p/q, t) \geq c/d\} \tag{1}$$

is rational, and the infimum is achieved. Also, since $R$ is locally constant from the right at rational points, it suffices to compute the infimum over rational $t$. So consider some $t = u/v$ (in lowest terms) such that $R(w;\ p/q, u/v) \geq c/d$. In fact, let $W$ be a $XY$ word of type $(q, v)$ for which $R(w;\ p/q, u/v) = \mathrm{rot}^{\sim}_{W}(w)$. After some cyclic permutation, we can write

$$W = Y^{t_1} X Y^{t_2} X Y^{t_3} X \cdots Y^{t_q} X,$$

where $t_i \geq 0$ and $\sum_{i=1}^{q} t_i = v$. Our goal is to then minimize $u/v$ over all such possible $XY$-words $W$.

After some circular permutation (which does not affect $R$), we may also assume without loss of generality that $w$ is of the form

$$w = b^{\beta_n} a^{\alpha_n} \cdots b^{\beta_2} a^{\alpha_2} b^{\beta_1} a^{\alpha_1},$$

where $\alpha_i, \beta_i > 0$. Also, assume that equality is achieved in (1) for $u/v$ (i.e., $R(w;\ p/q, t) = c/d$). Thus, by construction, the action of $w$ on $W$, defined via its action on $\mathbb{Z}$, is periodic with a period $d$, and a typical periodic orbit begins at $W_1 = Y$.

We fix some notation and try to analyze the action of each maximal string of $a$ or $b$ in $w$ on $W$ by inspecting its action on $\mathbb{Z}$. Note that, for

$$\tilde{s}_i = a^{\alpha_i} b^{\beta_{i-1}} a^{\alpha_{i-1}} \cdots b^{\beta_1} a^{\alpha_1}(1),$$

the $\tilde{s}_i$th letter in $W^\infty$ is always $X$. Let $s_i$ be the index modulo $q$ so that $W^\infty_{\tilde{s}_i}$ is the $s_i$th $X$ in $W$ (see Figure 3). Thus, for a periodic orbit starting at $W_1 = Y$, the string $b^{\beta_i}$ is applied to the $s_i$th $X$.

Then, by definition, $b^{\beta_i}(\tilde{s}_i)$ is the least number such that the sequence $W_{\tilde{s}_i}$, $W_{\tilde{s}_i+1}, \ldots, W_{b^{\beta_i}(\tilde{s}_i)}$ contains exactly $u\beta_i + 1$ $Y$s. Let $l_i$ denote the number of $X$s in the sequence $W_{\tilde{s}_i}(= X), W_{\tilde{s}_i+1}, \ldots, W_{b^{\beta_i}(\tilde{s}_i)}(= Y)$ (see Figure 4). Thus $l_i$ is the smallest number such that

$$t_{s_i+1} + t_{s_i+2} + \cdots + t_{s_i+l_i+1} \geq u\beta_i + 1. \tag{2}$$

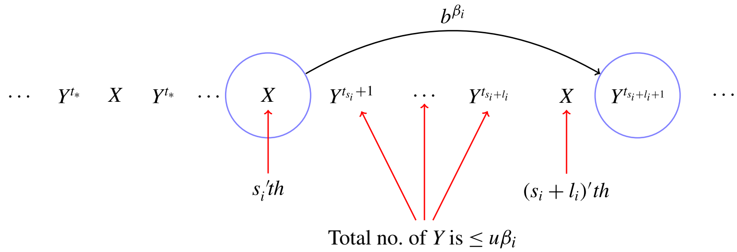


FIGURE 4. Action of $b^{\beta_i}$.

In other words, $l_i$ is the biggest number such that

$$t_{s_i+1} + t_{s_i+2} + \cdots + t_{s_i+l_i} \leq u\beta_i. \tag{3}$$

The purpose of rewriting this inequality was to make it homogeneous. Even if equality does not occur in (1), the inequality in (2) still holds true. The only difference is that $l_i$ does not necessarily have to be the smallest number; however, it does have to satisfy other constraints which we now describe.

We write $w^d$ as

$$w^d = b^{\beta_k} a^{\alpha_k} b^{\beta_{k-1}} a^{\alpha_{k-1}} \cdots b^{\beta_1} a^{\alpha_1}$$

and instead of considering the action of $w$ on $W$ with a period $d$, assume that $w^d$ acts on $W^c$ by its action on $\mathbb{Z}$. Then the maximal $a$-strings and $b$-strings in $w^d$ all together cover exactly the total number of $X$s (and $Y$s) in $W^c$. For a similar reason, we know that intervals of the form of $(W_j, W_{a^{\alpha_i}(j)})$ enclose precisely $p\alpha_i + 1$ $X$s. Thus we get the equality

$$\sum_{i=1}^{k} (l_i + (\alpha_i p + 1)) = cq.$$

Note that here the $\alpha_i$ are periodic as a function of $i$, with a period $k/d = n$ but, in general, the $l_i$ are not periodic in $i$. We can also give a formula for $s_i$ by counting the number of $X$s covered.

$$s_i = \sum_{j=1}^{i} (\alpha_j p + 1) + \sum_{j=1}^{i-1} l_j.$$

Thus, we have formulated our minimization problem as a homogeneous linear integral equation subject to a finite number of integral linear constraints. Because of homogeneity, it has a solution in integers if and only if it has a solution in rational numbers and, consequently, we can normalize the whole problem by rescaling to $v = 1$. Our algorithm is then as follows.

*Step 1.* Replacing $w$ by a cyclic permutation if necessary, write $w^d$ in the form $w^d = b^{\beta_k} a^{\alpha_k} \cdots b^{\beta_1} a^{\alpha_1}$.

*Step 2.* Enumerate all non-negative integral solutions to

$$\sum_{i=1}^{k} l_i = cq - \sum_{i=1}^{k} (\alpha_i p + 1).$$

*Step 3.* For each such solution set $(l_1, \ldots, l_k)$, define

$$s_i = \sum_{j=1}^{i}(\alpha_j p + 1) + \sum_{j=1}^{i-1} l_j.$$

*Step 4.* Find the smallest $u$ which satisfies the system of inequalities

$$\begin{cases} \sum_{i=1}^{q} t_i = 1, \\ t_i \geq 0 \quad \text{for all } i, \\ t_{s_i+1} + t_{s_i+2} + \cdots + t_{s_i+l_i} \leq u\beta_i \quad \text{for all } 1 \leq i \leq k \text{ (indices taken mod } q). \end{cases}$$

*Step 5.* Find the smallest $u$ over all solution sets $(l_1, \ldots, l_k)$.

The solution to this algorithm is necessarily rational and gives the minimal $t$ such that $R(w;\, p/q, t) \geq c/d$. Also, if equality is achieved, then clearly $R(w;\, p/q, u) = c/d$, and thus the theorem is proved. □

## 3. *A formula for fringe lengths*

In this section we will apply the stairstep algorithm to the computation of fringe lengths. The key idea is that in this special case, the equation

$$\sum_{i=1}^{k} l_i = cq - \sum_{i=1}^{k}(\alpha_i p + 1)$$

has a *unique* non-negative integral solution. This, in turn, reduces the last step of the algorithm to the solution of a *single* linear programming problem, rather than a system of many (exponentially increasing in number) inequalities.

### 3.1. *Proof of fringe formula 1.3.*

We now begin the proof of the fringe formula. This takes several steps and requires a careful analysis of the stairstep algorithm. We therefore adhere to the notation in §2.3. After cyclically permuting $w$, if necessary, we write $w$ in the form

$$w = b^{\beta_n} a^{\alpha_n} \cdots b^{\beta_1} a^{\alpha_1}.$$

#### 3.1.1. *Finding the optimal partition.*

First, note that by Theorem 1.2, it is enough to find the minimum $t$ such that

$$R(w;\, p/q, t) = \frac{h_a p + h_b q}{q}.$$

Thus, to apply the stairstep algorithm (2.3), we are going to fix $c/d = (h_a p + h_b q)/q$ where $c/d$ is the reduced form. Let us denote the $gcd$ of $h_a$ and $q$ by $g$ so that we have

$$c = \frac{h_a p + h_b q}{g} \quad \text{and} \quad d = \frac{q}{g},$$

since $(p, q) = 1$. Further, using $h_a = h'g$ and $q = q'g$, we rewrite the above equations as

$$c = h'p + h_b q' \quad \text{and} \quad d = q'.$$

Thus, step 1 of our algorithm becomes

$$w^{q'} = b^{\beta_{nq'}} a^{\alpha_{nq'}} \cdots b^{\beta_1} a^{\alpha_1},$$

where clearly $\alpha_i$, $\beta_i$ are periodic as functions of $i$ with period $n$.

Similarly, step 2 of our algorithm transforms to

$$l_1 + \cdots + l_{q'\cdot n} = \underbrace{\frac{h_a \cdot p + h_b \cdot q}{g}}_{=c} \cdot q - \underbrace{q' \cdot h_a}_{=\sum_{i=1}^{nq'} \alpha_i} \cdot p - q' \cdot n,$$

i.e.,

$$l_1 + \cdots + l_{nq'} = h_b \cdot qq' - nq' \tag{4}$$

and the equations in step 4 to find the minimum solution $u$, become

$$\sum_{i=1}^{q} t_i = 1, \tag{5}$$

$$t_i \geq 0 \quad \text{for all } i, \tag{6}$$

$$t_{s_i+1} + t_{s_i+2} + \cdots + t_{s_i+l_i} \leq \beta_i u \quad \text{for all } 1 \leq i \leq nq', \tag{7}$$

where indices are taken (mod $q$). Now if any of the $l_i$ is greater than or equal to $q\beta_i$, then the indices on the left-hand side of equation (7) cycle through all of 1 through $q$ a total of $\beta_i$ times. Then using (5), we get that

$$\beta_i = \beta_i \sum_{1}^{q} t_i \leq t_{s_i+1} + t_{s_i+2} + \cdots + t_{s_i+l_i} \leq \beta_i u,$$

implying $u \geq 1$, which is clearly not the optimal solution. Hence, for the minimal solution $u$, we must have

$$l_i \leq q\beta_i - 1 \quad \text{for all } 1 \leq i \leq nq'.$$

Summing up all of these inequalities, we get that

$$\sum_{i=1}^{nq'} l_i \leq q \sum_{i=1}^{nq'} \beta_i - nq' = qq'h_b - nq'.$$

But, on the other hand, by step 2, equality is indeed achieved in the inequality above and hence

$$l_i = q\beta_i - 1 \quad \text{for all } 1 \leq i \leq nq' \tag{8}$$

is the *unique* non-negative integral solution to the partition problem in step 2. As mentioned before, this means that from now on we only need to deal with a single linear programming problem, which is formulated more precisely in the next section.

3.1.2. *A linear programming problem.* With the specific values of $l_i$ found above, we can transform equations (5)–(7) as follows. Note that for $l_i = q\beta_i - 1$, the set of indices $s_i + 1, s_i + 2, \dots, s_i + l_i$ cycle through all of the values $1, 2, \dots, q$ a total of $\beta_i$ times, except one of them, namely $s_i \pmod q$, which appears $\beta_i - 1$ times. Then we can rewrite (7) as

$$\beta_i\left(\sum_{j=1}^{q} t_j\right) - t_{s_i} \le \beta_i u \quad \text{for all } 1 \le i \le nq',$$

i.e.,

$$\frac{t_{s_i}}{\beta_i} \ge 1 - u \quad \text{for all } 1 \le i \le nq'.$$

Observe that in the above equation, the $\beta_i$ are periodic with a period $n$ whereas the $s_i$ are well defined modulo $q$ (since the $t_i$ have period $q$), which is usually much bigger than $n$. Then for the purpose of finding a value of $u$ which satisfies the system of equations (5)–(7), it will be enough to consider the indices $i$ for which $\beta_i$ is maximum for the same value of $s_i$.

To make the statement more precise, we introduce the following notation. Let the set of indices $\Lambda$ be defined by

$$\Lambda = \left\{ i \;\middle|\; \beta_i = \max_{\substack{s_j = s_i \\ 1 \le j \le nq'}} \beta_j \right\}.$$

Then the first thing to note is that the set of numbers $\{s_i\}_{i \in \Lambda}$ are all distinct. Next, recall that we are in fact trying to find the fringe length, which is $1 - t$, where $t$ is the solution to the stairstep algorithm. So, with a simple change of variable, our algorithm becomes the following *linear programming problem*:

$$\text{Find maximum of} \quad \min_{i \in \Lambda}\left\{\frac{1}{\beta_i} t_{s_i}\right\},$$

$$\text{subject to} \quad \sum_{i \in \Lambda} t_{s_i} \le 1,\; t_{s_i} \ge 0 \quad \text{for all } i.$$

But since we are trying to find the maximum, we may as well assume that $\sum_{i \in \Lambda} t_{s_i} = 1$ and $t_k = 0$ if $k \ne s_i$ for some $i \in \Lambda$. Then, by a theorem of Kaplan [**9**], we get that the optimal solution occurs when for all $i \in \Lambda$, the number $t_{s_i}/\beta_i$ equals some constant $T$ independent of $i$. To find $T$, observe that

$$\frac{t_{s_i}}{\beta_i} = T \Rightarrow \sum_{i \in \Lambda} \beta_i T = 1 \Rightarrow T = \frac{1}{\sum_{i \in \Lambda} \beta_i}.$$

Thus the optimal solution to the linear programming problem, which is also the required fringe length, is given by

$$\mathrm{fr}_w(p/q) = \frac{1}{\sum_{i \in \Lambda} \beta_i}. \tag{9}$$

So all that remains is to figure out what the set of indices $\Lambda$ looks like. In the rest of this section we try to characterize $\Lambda$ and prove the fringe formula 1.3.

3.1.3. *Reduction to combinatorics.* It is clear from the definition that to figure out the set $\Lambda$ we need to find out exactly when two of the $s_i$ are equal as $i$ ranges from 1 to $nq'$. Recall that the indices $s_i$ are taken modulo $q$. Using the optimal partition, we get that

$$s_i + l_i = \sum_{j=1}^{i} (p\alpha_j + 1 + q\beta_j - 1)$$

and hence

$$s_I = s_J \Leftrightarrow \sum_{j=1}^{I} \alpha_j \equiv \sum_{j=1}^{J} \alpha_j \pmod{q},$$

since $l_I \equiv l_J \pmod{q}$. Thus the elements of $\Lambda$ are in bijective correspondence with the number of residue classes modulo $q$ in the following set of numbers.

$$\begin{gathered} A_1 = \alpha_1 \\ A_2 = \alpha_1 + \alpha_2 \\ A_3 = \alpha_1 + \alpha_2 + \alpha_3 \\ A_4 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \vdots \\ A_{nq'} = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{nq'}. \end{gathered}$$

So we can rewrite the formula for the set $\Lambda$ as

$$\Lambda = \left\{ i \;\middle|\; \beta_i = \max_{\substack{A_j \equiv A_i \pmod{q} \\ 1 \le j \le nq'}} \beta_j \right\}.$$

Note that $A_n = h_a$ and the $\alpha_i$ are periodic with period $n$. So we have, $A_{n+i} = A_i + h_a$ or, in other words, the collection of numbers $A_1, A_2, \ldots, A_{nq'}$ is nothing but a union of disjoint translates of the collection $(A_1, A_2, \ldots, A_n)$ by $0, h_a, 2h_a, \ldots, (q'-1)h_a$.

Let us refer to the $n$-tuple $(A_1, A_2, \ldots, A_n)$ as the first '$n$-block'. Similarly, the $h_a$-translate of the first $n$-block is referred to as the second $n$-block, and so on. Note that $q'h_a = h'q$, so the $q'h_a$-translate of the first $n$-block is identical to itself modulo $q$. Hence we may think of translation by $(q'-1)h_a$ as translation by $-h_a$.

Next, we claim that

CLAIM. *The numbers* $0, h_a, 2h_a, \ldots, (q'-1)h_a$ *are all distinct modulo* $q$.

*Proof.* If $q$ divides the difference between any two such numbers, say $mh_a$, then $q' \mid mh' \Rightarrow q' \mid m \Rightarrow m \geq q'$, which is a contradiction. □

In fact since $h'$ is invertible modulo $q$, the set of numbers $\{0, h_a, \ldots, (q'-1)h_a\}$ is the same as $\{0, g, 2g, \ldots, (q'-1)g\}$ modulo $q$. Thus, to determine the congruence classes in the collection $A_1, A_2, \ldots, A_{nq'}$, it is enough to find out which $n$-blocks overlap with the first $n$-block. Note that translating an $n$-block by $h_a (= h'g)$ takes it off itself entirely, so the *only* translates of an $n$-block that could overlap with itself are the translates by $ig$ for $|i| < h'$ (see Figure 5).

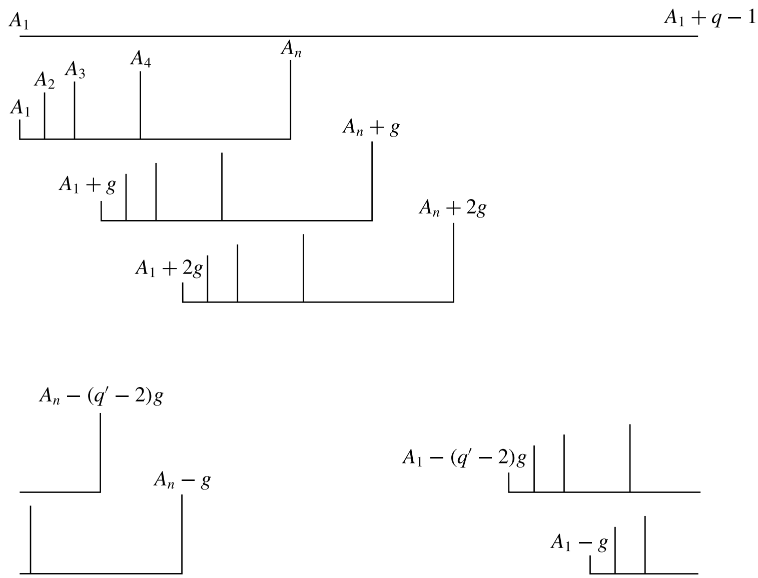


FIGURE 5. Translates of the first $n$-block.

Finally, observe that if we start with the the $n$-block given by $(A_1 + g, A_2 + g, \ldots, A_n + g)$ instead, we get overlaps at the same multiple of $g$ as the first $n$-block; but translated by $g$. Thus, starting from $A_1$, if we divide the residue class of $q$ into a total of $q'$ number of $g$-sized groups, then each $\beta_i$ appears the same number of times in each group and the overlaps appear at the same places translated by multiples of $g$. Hence to calculate the sum of $\max\{\beta_i\}$ over all residue classes, it is enough to calculate it for the residue classes which appear among $A_1, A_1 + 1, A_1 + 2, \ldots,$ up to $A_1 + (g - 1)$ and then multiply the result by $q'$.

Let us summarize the results we have found so far in the form of an algorithm.

*Step 1.* Write down $A_1, A_2, \ldots, A_n$ where $A_i = \alpha_1 + \cdots + \alpha_i$.

*Step 2.* For each $0 \le i \le g - 1$, let $\mathfrak{B}_i$ be defined as

$$\mathfrak{B}_i = \max\{\beta_{k+mg} \mid A_{k+mg} \equiv A_1 + i \pmod q \text{ where } -h' < m < h',\ 1 \le k \le n\}.$$

Note that in the case $q' < h'$, we replace $h'$ with $q'$ in the above definition.

*Step 3.* Let $S$ be the sum of the $\mathfrak{B}_i$ for $0 \le i \le g - 1$. Then the fringe length is given by

$$\mathrm{fr}_w(p/q) = \frac{1}{q'S}. \tag{10}$$

To finish the proof, define $\sigma_w(g) := S/g$ and note that by the structure of the algorithm, $\sigma_w(g)$ depends only on $g = gcd(q, h_a)$ and the word $w$. As a corollary, we also get the following remarkable consequence.

COROLLARY 3.1. *The fringe length* does not *depend on p.*

That is, the fringes are ‘periodic’ on every scale. In §5 we elaborate on this phenomenon in a particular example, and discuss possible generalizations.

## 4. *Examples and special cases*

In this section, we give some examples to illustrate the complexity of the function $\sigma$ in general, and in the special case that $h_a(w)$ is prime. Let us first prove the following.

THEOREM 4.1. ($\sigma$-inequality) *Suppose $w = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots a^{\alpha_n} b^{\beta_n}$. Then the function $\sigma_w(g)$ satisfies the inequality*

$$\frac{h_b}{h_a} \le \sigma_w(g) \le \max_{1\le i\le n} \beta_i,$$

*where the first equality is achieved in the case when $h_a$ divides $q$ and the second equality occurs when $(q, h_a) = 1$.*

*Proof.* For the first inequality, recall the numbers $A_1, A_2, \ldots, A_{nq'}$ from last section. Note that the fact that $h_a \cdot q' = h' \cdot q$ tells us that there are at most $h'$ elements in each residue class modulo $q$ among $A_1, \ldots, A_{nq'}$. Thus

$$\sum_{i=1}^{nq'} t_{s_i} \le h' \cdot \sum_{i\in\Lambda} t_{s_i} \le h' \cdot \sum_{i=1}^{q} t_i = h'.$$

On the other hand, adding all the $nq'$ inequalities in (7), and using $l_i = q\beta_i - 1$, we get that

$$u \cdot \sum_{i=1}^{nq'} \beta_i \ge \sum_{i=1}^{nq'} \left( \beta_i \sum_{j=1}^{q} t_j - t_{s_i} \right) = \sum_{i=1}^{nq'} \beta_i - \sum_{i=1}^{nq'} t_{s_i} \ge \sum_{i=1}^{nq'} \beta_i - h'$$

$$\text{and } u \ge 1 - \frac{h'}{h_b \cdot q'} = 1 - \frac{h_a}{h_b q}.$$

Hence, for the minimal $u$ giving the fringe length we get that

$$\sigma_w(g) \ge \frac{h_b}{h_a}.$$

For the second inequality, observe that, by definition,

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g) q} = \frac{1}{\sum_{i\in\Lambda} \beta_i} \ge \frac{1}{|\Lambda| \cdot \max_{i\in\Lambda} \beta_i} \ge \frac{1}{q \cdot \max_{i\in\Lambda} \beta_i},$$

since the number of elements in $\Lambda$ is at most the number of residue classes modulo $q$. Hence

$$\sigma_w(g) \le \max_{i\in\Lambda} \beta_i \le \max_{1\le i\le n} \beta_i.$$

We will finish the proof by showing that equality is indeed achieved in the following special cases.

*Case 1: $h_a \mid q$.* In this case $h' = 1$. Hence all the $s_i$ are distinct.

Consider the specific example where $t_{s_i} = \beta_i/(h_b q')$ for all $i$ and the rest of the $t_i$ are zero. Then we have

$$\beta_i \cdot u \ge \sum_{j\ne i} \frac{\beta_j}{h_b q'} \cdot \beta_i + \frac{\beta_i}{h_b q'} \cdot (\beta_i - 1) = \beta_i \cdot \frac{h_b q'}{h_b q'} - \frac{\beta_i}{h_b q'} \Rightarrow u \ge 1 - \frac{1}{h_b q'}.$$

So the minimum $u_0$ which gives a solution to (5)–(7) is $1 - 1/(q' h_b) = 1 - h_a/(h_b q)$. Thus equality is achieved in the first part of Theorem 1.4.

We can give a second proof of this same fact using the algorithm developed in the last section. Since $h_a \mid q$, the $gcd$ of $h_a$ and $q$ is $h_a$. So any $g$-translate of the $n$-block is disjoint from itself. Hence $S = h_b$, giving the same formula as above.

*Case 2:* $gcd(h_a, q) = 1$. In this situation, $g = 1$. Hence $c = h_a \cdot p + h_b \cdot q$ and $d = q$, since $q = q'$.

Let $W = Y^{t_1} X Y^{t_2} \cdots Y^{t_q} X$ as in the proof of Theorem 2.3. Since $w$ now has a periodic orbit of period exactly $q$, we get that any $b$-string starting on adjacent $X$ must land in adjacent $Y^*$ strings. Thus the constraints of the linear programming problem are invariant under permutation of the variable $t_i$, and by convexity, extrema are achieved when all the $t_i$ are equal. But then we get

$$q \cdot t_i = 1 \Rightarrow t_i = \frac{1}{q}$$

and

$$\beta_i u \geq l_i \cdot t_i = \frac{(q\beta_i - 1)}{q} \Rightarrow u \geq 1 - \frac{1}{q\beta_i} \quad \text{for all } 1 \leq i \leq nq.$$

Hence the minimum $u$ which gives a solution to the system of equation is given by

$$u = 1 - \frac{1}{q \cdot \max_{1 \leq i \leq n}\{\beta_i\}}.$$

Observing that equality is indeed achieved in case of the word $(XY^{\max\{\beta_i\}})^q$, we get equality in the second part of Theorem 1.4.

Again, we can give a much simpler proof of this result using the algorithm in the last section. In this case, we have $g = 1$ so that $q = q'$. So $S$ is the maximum of all the $\beta_i$ which correspond to any $A_i$ that is a translate of $A_1$ by one of $-h_a, -h_a + 1, \ldots, 0, \ldots, h_a - 1, h_a$ (i.e., all of the $A_i$). Thus $S = \sigma_w(g) = \max_{1 \leq i \leq n}\{\beta_i\}$, since $g = 1$. □

COROLLARY 4.2. *If $h_a$ is a prime number then*

$$\mathrm{fr}_w(p/q) = \begin{cases} \dfrac{h_a}{q \cdot h_b} & \text{if } h_a \mid q, \\[2ex] \dfrac{1}{q \cdot \max_{1 \leq i \leq n} \beta_i} & \text{if } h_a \nmid q. \end{cases}$$

*Remark 4.3.* The function $\sigma_w(g)$ depends on $g = gcd(h_a, q)$ in a complicated way when $h_a$ is not prime, as we can see from Table 1.

*Example 4.4.* Let us consider the case of the word $w = abaab$. By Corollary 4.2, the left fringe lengths are given by

$$\mathrm{fr}_w(p/q) = \begin{cases} \dfrac{3}{2q} & \text{when } 3 \mid q, \\[2ex] \dfrac{1}{q} & \text{when } 3 \nmid q, \end{cases}$$

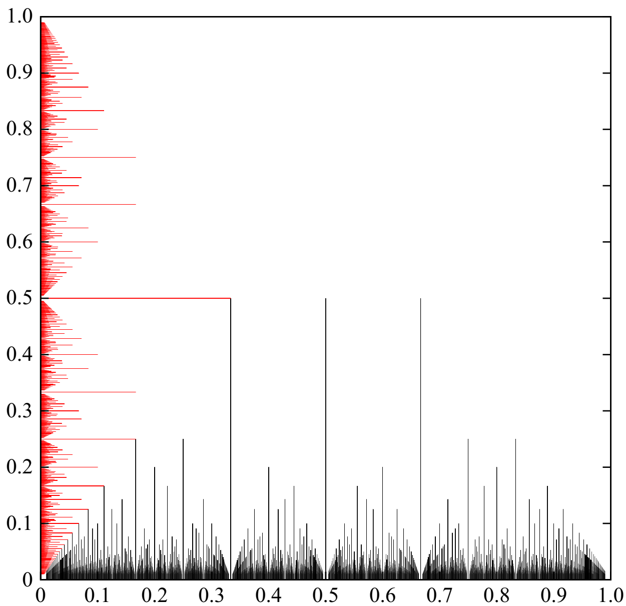


FIGURE 6. Plot of the fringes of *abaab*, $q = 1$ to 100.

TABLE 1. Values of $\sigma_w(g)$ for different $w$ and $g$.

| Word $h_a = 6$ | $h_b$ | $p/q = 1/5$ $g = 1$ | $p/q = 1/2$ $g = 2$ | $p/q = 1/3$ $g = 3$ | $p/q = 1/6$ $g = 6$ |
|---|---|---|---|---|---|
| *aaabaaabbbb* | 5 | 4 | 5/2 | 4/3 | 5/6 |
| *abaabaaabbbb* | 6 | 4 | 5/2 | 5/3 | 1 |
| *abbaabaaabbbb* | 7 | 4 | 3 | 2 | 7/6 |
| *abbbaabaaabbbb* | 8 | 4 | 7/2 | 4/3 | 7/3 |
| *abbbababaaabbbb* | 9 | 4 | 7/2 | 8/3 | 3/2 |
| *abbbaabbaaabbbb* | 9 | 4 | 7/3 | 7/3 | 3/2 |
| *abbbababbaaabbbb* | 10 | 4 | 7/2 | 8/3 | 5/3 |

and the right fringe lengths are given by

$$\mathrm{fr}_w(p/q) = \begin{cases} \dfrac{2}{3q} & \text{when } q \text{ is even,} \\ \dfrac{1}{2q} & \text{when } q \text{ is odd.} \end{cases}$$

The cases when $3 \nmid q$ and $2 \nmid q$ were also discussed in [**4**, p. 18].

We finish this section by giving a fringe plot for both sides for the word $w = abaab$. Let us put the origin at the point $(r = 1, s = 1)$ and the point $(r = 0, s = 0)$ be depicted as $(1, 1)$. Then we have Figure 6.

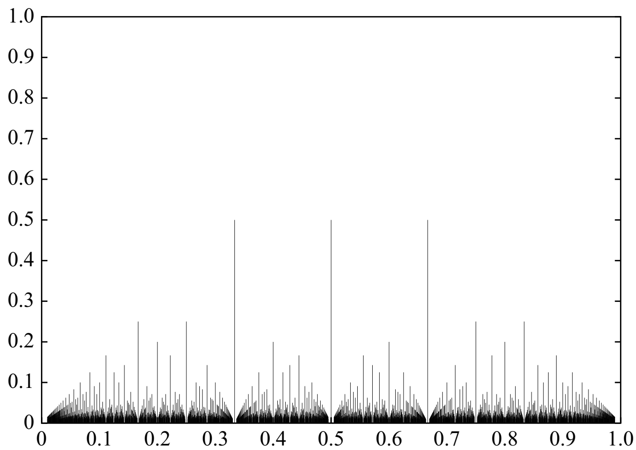


FIGURE 7. Plot of left Fringe, $q = 1$ to 100.

5. *Projective self similarity*

In her paper, Gordenko shows that the the Ziggurat of the word $w = ab$ is self-similar under two projective transformation [**7**, Theorem 4]. In this section we show that similar transformations exist in case of the word $w = abaab$, which gives a different way of looking at the fringe formula.

First, let us look at the self-similarities of the left fringe. Figure 7 shows a plot of the fringe lengths where the $x$-axis is the value of $\mathrm{rot}^{\sim}(a)$ and the $y$-axis is value of $\mathrm{fr}_{abaab}(x)$. Thus for $x = p/q$ we have $\mathrm{fr}_{abaab}(x)$ defined as in Example (4.4). We will drop the subscript $abaab$ for the next part.

We prove that the unit interval can be decomposed into some finite number of intervals $\Delta_i$ such that there exists a further decomposition of each $\Delta_i$ into a disjoint union of subintervals $I_{i,j}$ with the graph of $\mathrm{fr}(x)$ on each of $I_{i,j}$ being similar to that on some $\Delta_{k(i,j)}$ under projective linear transformations.

THEOREM 5.1. *Let* $\Delta_1 = (0, 1/3)$, $\Delta_2 = (1/3, 1/2)$, $\Delta_3 = (1/2, 2/3)$ *and* $\Delta_4 = (2/3, 1)$. *Then we have the following decomposition into* $I_{i,j}$ *and transformations* $T_{i,j}$.

$$I_{1,1} = (0, 1/4), \qquad T_{1,1}(I_{1,1}) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 = [0, 1],$$
$$T_{1,1}(x, y) = \left(\frac{x}{1-3x}, \frac{y}{1-3x}\right),$$
$$I_{1,2} = (1/4, 1/3), \qquad T_{1,2}(I_{1,2}) = \Delta_1,$$
$$T_{1,2}(x, y) = \left(\frac{4x-1}{9x-2}, \frac{y}{9x-2}\right),$$
$$I_{2,1} = (1/3, 1/2), \qquad T_{2,1}(I_{2,1}) = \Delta_1,$$
$$T_{2,1}(x, y) = \left(\frac{1-2x}{2-3x}, \frac{y}{2-3x}\right).$$

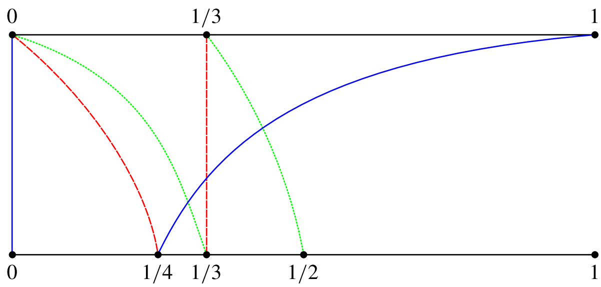


FIGURE 8. Intervals of projective self-similarity in case of $w = abaab$.

*Since the graph is clearly symmetric about $x = 1/2$, similar decomposition exists for $\Delta_3$ and $\Delta_4$ (see Figure 8).*

*Proof.* For each of the transformations, note that the denominator of the image of $p/q$ has the same greatest common divisor, $h_a$, as $q$. Also, in each case, the numerator and denominator are coprime. The proof then follows easily by checking the length of images in each case. □

We thus note that $\Delta_1$ contains all the information necessary to determine the fringe dynamics. In fact, for $h_a$ prime the following similarity result always holds.

THEOREM 5.2. *Let $\Delta_1 = (0, 1/h_a)$ where $h_a$ is a prime number. Then we can decompose $\Delta_1$ into $I_{i,j}$ and find transformations $T_{i,j}$ as follows.*

$$I_{1,1} = (0, 1/(h_a + 1)), \qquad T_{1,1}(I_{1,1}) = [0, 1],$$

$$T_{1,1}(x, y) = \left(\frac{x}{1 - h_a x}, \frac{y}{1 - h_a x}\right),$$

$$I_{1,2} = (1/(h_a + 1), 1/h_a), \qquad T_{1,2}(I_{1,2}) = \Delta_1,$$

$$T_{1,2}(x, y) = \left(\frac{(h_a + 1)x - 1}{h_a^2 x - (h_a - 1)}, \frac{y}{h_a^2 x - (h_a - 1)}\right).$$

It is also easy to prove, in the case of prime $h_a$, that the plot on $\Delta = [(h_a - 1)/2h_a, \frac{1}{2}]$ is similar to $\Delta_1$ under the transformation

$$T(x, y) = \left(\frac{2 - 4x}{(h_a + 1) - 2h_a x}, \frac{2y}{(h_a + 1) - 2h_a x}\right).$$

Note that, in the case of $h_a = 3$, we have $(h_a - 1)/2h_a = 1/h_a$, which explains Theorem 5.1.

*Acknowledgements.* I would like to thank Anna Gordenko for sharing her preprint [**7**], which directly inspired the main problem studied in this paper, and Victor Kleptsyn, Alden Walker and Jonathan Bowden for some useful discussions. I would also like to thank Clark Butler, Amie Wilkinson, Paul Apisa and the anonymous referee for several helpful comments. Finally, I would like to thank Danny Calegari, my advisor, for his continued support and guidance, as well as for the extensive comments and corrections on this paper; and for providing the thanksgiving turkey.

## REFERENCES

[**1**] S. Agnihotri and C. Woodward. Eigenvalues of products of unitary matrices and quantum Schubert calculus. *Math. Res. Lett.* **5**(6) (1998), 817–836.

[**2**] P. Belkale. Local systems on $P^1 - S$ for $S$ a finite set. *Compos. Math.* **129**(1) (2001), 67–86.

[**3**] M. Bucher, R. Frigerio and T. Hartnick. A note on semi-conjugacy for circle actions. *Preprint*, 2014, arXiv:1410.8350.

[**4**] D. Calegari and A. Walker. Ziggurats and rotation numbers. *J. Mod. Dyn.* **5**(4) (2011), 711–746.

[**5**] R. E. Ecke, J. Doyne Farmer and D. K. Umberger. Scaling of the Arnol'd tongues. *Nonlinearity* **2**(2) (1989), 175–196.

[**6**] É. Ghys. Groups acting on the circle. *Enseign. Math.* (2) **47** (2001), 329–407.

[**7**] A. Gordenko. Self-similarity of Jankins–Neumann ziggurat. *Preprint*, 2015, arXiv:1503.03114.

[**8**] M. Jankins and W. Neumann. Rotation numbers of products of circle homeomorphisms. *Math. Ann.* **271**(3) (1985), 381–400.

[**9**] S. Kaplan. Application of programs with maximin objective functions to problems of optimal resource allocation. *Oper. Res.* **22**(4) (1974), 802–807.

[**10**] R. Naimi. Foliations transverse to fibers of Seifert manifolds. *Comment. Math. Helv.* **69**(1) (1994), 155–162.