

A TOPOLOGICAL PROOF THAT O_2 IS 2-MCFL

SUBHADIP CHOWDHURY

ABSTRACT. We give a new proof of Salvati's theorem that the group language O_2 is 2 multiple context free. Unlike Salvati's proof, our arguments do not use any idea specific to two-dimensions. This raises the possibility that the argument might generalize to O_n .

CONTENTS

1. Introduction	1
2. Acknowledgement	2
3. A generalization of Salvati's Theorem in 2-dimension	2
4. Proof of Salvati's Theorem	10
5. The case of 3 dimension and higher	13
References	14

1. INTRODUCTION

In a recent paper[3], Sylvain Salvati proved that the language

$$O_2 = \{w \in \{a, b, A, B\}^* \mid |w|_a = |w|_A \wedge |w|_b = |w|_B\}$$

and hence the rationally equivalent language

$$\text{MIX} = \{w \in \{a, b, c\}^* \mid |w|_a = |w|_b = |w|_c\}$$

are 2-Multiple Context Free Languages as follows. Consider the grammar

$$G = (\Omega, \mathcal{A}, R, S)$$

where $\Omega = (\{S; \text{Inv}\}, \rho)$ with $\rho(S) = 1$ and $\rho(\text{Inv}) = 2$; $\mathcal{A} = \{a, A, b, B\}$ and R consists of the following rules:

- (1) $S(x_1x_2) \leftarrow \text{Inv}(x_1, x_2)$
- (2) $\text{Inv}(t_1, t_2) \leftarrow \text{Inv}(x_1, x_2)$ where $t_1t_2 \in \text{perm}(x_1x_2aA) \cup \text{perm}(x_1x_2bB)$
- (3) $\text{Inv}(t_1, t_2) \leftarrow \text{Inv}(x_1, x_2), \text{Inv}(y_1, y_2)$ where $t_1t_2 \in \text{perm}(x_1x_2y_1y_2)$
- (4) $\text{Inv}(\epsilon, \epsilon)$

Note that by construction it is easy to prove that the language $L = \{w \mid S(w) \text{ is derivable in } G\}$ generated by G , is a subset of O_2 . Using ideas specific to two-dimensional geometry, e.g. complex exponential function, Salvati then proves that

Theorem 4.2 (Salvati). *If $w_1w_2 \in O_2$, then $\text{Inv}(w_1, w_2)$ is derivable and hence w_1w_2 will be in the language $L = \{w \mid S(w) \text{ is derivable in } G\}$ generated by G .*

Date: November 1, 2017.

Theorem 4.2 along with the previous description of the grammar G thus proves that $L = O_2$. In this paper, we give a different proof of the same theorem using ideas from homology theory that are not specific to two dimensions. This raises the possibility that the proof might be generalized to higher dimensional cases, thus shedding light on the (still open) question whether O_n is a n -MCFL.

2. ACKNOWLEDGEMENT

I would like to thank Danny Calegari, my advisor, for sharing with me Salvati's paper [3] which directly inspired the main result of this paper; and for his continued support and guidance, as well as for the extensive comments and corrections on this paper. I would also like to thank Mark-Jan Nederhof for sharing his paper [2] and other related works to this problem and Greg Kobele for some useful discussion and comments.

3. A GENERALIZATION OF SALVATI'S THEOREM IN 2-DIMENSION

A word in O_2 corresponds uniquely to a closed path in the integer lattice \mathbb{Z}^2 as follows. a and b correspond to \rightarrow and \uparrow , and A, B to \leftarrow and \downarrow respectively. e.g. the word $abbAbaBaBBBAbA \in O_2$ corresponds to the path in figure 1.

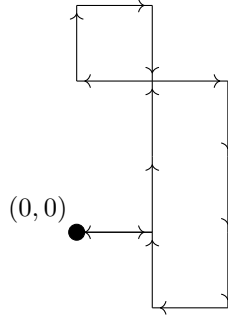


FIGURE 1. Lattice path corresponding to the word $abbAbaBaBBBAbA$

We prove a purely topological result about closed curves on the plane (not necessarily simple) which, as a corollary, gives us theorem 4.2.

Consider a closed (oriented) loop K in \mathbb{R}^2 given by

$$\varphi : S^1 \rightarrow \mathbb{R}^2, \quad \text{Image}(\varphi) =: K$$

It makes no difference to set the domain of φ equal to $[0, 1]$ and assume $\varphi(0) = \varphi(1)$. Let $p = \varphi(0), q = \varphi(1/2)$ and let r and s be two arbitrary points on K . Together the four points $\{p, q, r, s\}$ break up K into 4 (possibly degenerate) arcs K_1, K_2, K_3 and K_4 such that the starting point of K_{i+1} is the same as the ending point of K_i . Denote by \vec{v}_i the vector which is defined as

$$\vec{v}_i = \text{End point of } K_i - \text{Starting point of } K_i$$

The vectors \vec{v}_i satisfy

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 = 0$$

Our main technical result is the following.

Proposition 3.1. *Assume that φ is differentiable at p and q and $\varphi'(0)$ is not antiparallel to $\varphi'(1/2)$. Then there exists a pair of r and s on K different from $\{p, q\}$ such that the set of 4 vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ can be partitioned into two sets of 2 vectors each of which sum up to zero.*

The proof of Proposition 3.1 proceeds in multiple steps as follows. First we construct a 2 dimensional cell complex \mathcal{X} that parametrizes the choices for r and s along with a choice of partition for $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Then we define a function $f : \mathcal{X} \rightarrow \mathbb{R}^2$ with the property that a zero for f gives a choice of $\{r, s\}$ satisfying the conclusion of proposition 3.1. Finally using a homological argument, we show that f must have a zero.

Before moving on with the proof we make the following observations to simplify the proof. We claim that it suffices consider embedded K . Suppose the loop K has at least one self intersection i.e. it's not embedded. Then there exist two points $0 \leq t_1 < t_2 \leq 1$ such that $\varphi(t_1) = \varphi(t_2)$. Now depending on values of t_i , we must have one of the following cases (recall that p and q are distinct):

- (1) $t_1 = 0$ and $0 < t_2 < 1/2$

Then we remove $\varphi(0, t_2)$ of K to get a new knot K' . Clearly if K' satisfies the conclusion of proposition 3.1 then so does K . So we replace K with this new simplified knot K' .

- (2) $t_1 = 0$ and $1/2 < t_2 < 1$

In this case we remove $\varphi(t_2, 1)$ instead.

- (3) $t_2 = 1/2$ and $0 < t_1 < 1/2$ OR $t_1 = 1/2$ and $1/2 < t_2 < 1$

Similar to the above two cases.

- (4) $0 < t_1 < t_2 < 1/2$ OR $1/2 < t_1 < t_2 < 1$

Remove $\varphi(t_1, t_2)$.

- (5) $0 < t_1 < 1/2 < t_2 < 1$

Then choose $r = \varphi(t_1)$ and $s = \varphi(t_2)$ so that $\vec{v}_1 + \vec{v}_4 = \vec{v}_2 + \vec{v}_3 = 0$.

In conclusion, after necessary simplifications, without loss of generality we can assume that K is an embedded loop in \mathbb{R}^2 .

3.1. Construction of the 2 dimensional Cell Complex \mathcal{X} and the function f . In this section, we define the 2-complex \mathcal{X} whose points parametrizes choices of $\{r, s\} \neq \{p, q\}$ on K along with the particular choice of partition of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ into two sets of 2 vectors each. Note that the choice of partition also defines the function f on \mathcal{X} , but this is a bit complicated since it depends on the relative orders of $\{p, q, r, s\}$ on K . Now choosing r and s on K is equivalent to choosing two numbers x and y in the interval $[0, 1]$, where 0 and 1 are considered the same number. In what follows, we list all the possible 2-cells in \mathcal{X} in table 1; along with the 1-cells and 0-cells that appear as the boundary of the 2-cells and parametrize the cases when our choice is degenerate. This will take up next couple of pages. However, note that for our subsequent sections and to prove the main result, we will be using only some of these cells.

The 2-cells fall into three categories:

Case 1. $0 \leq x \leq 1/2 \leq y \leq 1$

Case 2. $0 \leq x \leq y \leq 1/2$

Case 3. $1/2 \leq x \leq y \leq 1$

In each case, we define f in such a way that $f(x, y) = 0$ implies existence of a partition of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ into two sets of two vectors each of which add up to zero.

Observe that if (p, r, s, q) appear in that particular order in K and $\vec{v}_2 = s - r = 0$ (which is equivalent to saying $\vec{v}_1 + \vec{v}_3 + \vec{v}_4 = 0$), then that means part of K from $\varphi(x)$ to $\varphi(y)$ makes a loop. But since we assumed K is embedded, those combination of vector(s) are never zero for any choice of x and y in case 2. So we can safely add the cells F and H in \mathcal{X} . Similarly, when p, q, r, s appear in that particular order in K we can consider the extra 2-cells J and L .

TABLE 1. The list of 2-cells

Name	picture of the cell	$f(x, y)$ on the cell	Name	picture of the cell	$f(x, y)$ on the cell
2-CELLS					
A		$\varphi(x) - \varphi(0) + \varphi(y) - \varphi(1/2) = r - p + s - q$	B		$\varphi(1/2) - \varphi(x) + \varphi(y) - \varphi(1/2) = q - r + s - q$
C		$\varphi(1/2) - \varphi(x) + \varphi(1) - \varphi(y) = q - r + p - s$	D		$\varphi(x) - \varphi(0) + \varphi(1) - \varphi(y) = r - p + p - s$
E		$\varphi(x) - \varphi(0) + \varphi(1/2) - \varphi(y) = r - p + q - s$	F		$\varphi(y) - \varphi(x) = s - r$
G		$\varphi(y) - \varphi(x) + \varphi(1) - \varphi(1/2) = s - r + p - q$	H		$\varphi(1/2) - \varphi(y) + \varphi(1) - \varphi(1/2) + \varphi(x) - \varphi(0) = \varphi(x) - \varphi(y) = r - p + q - s + p - q$
I		$\varphi(x) - \varphi(1/2) + \varphi(1) - \varphi(y) = p - s + r - q$	J		$\varphi(y) - \varphi(x) = s - r$
K		$\varphi(y) - \varphi(x) + \varphi(1/2) - \varphi(0) = q - p + s - r$	L		$\varphi(x) - \varphi(y) = q - p + r - q + p - s$
1-CELLS					
α		$\varphi(x) - \varphi(0) = r - p$	β		$\varphi(1/2) - \varphi(x) = q - r$

TABLE 1. (continued)

Name	picture of the cell	$f(x, y)$ on the cell	Name	picture of the cell	$f(x, y)$ on the cell
γ		$\varphi(y) - \varphi(1/2) = s - q$	δ		$\varphi(1) - \varphi(y) = p - s$
$\bar{\alpha}$		$\varphi(0) - \varphi(x) = p - r$	$\bar{\beta}$		$\varphi(x) - \varphi(1/2) = r - q$
$\bar{\gamma}$		$\varphi(1/2) - \varphi(y) = q - s$	$\bar{\delta}$		$\varphi(y) - \varphi(1) = s - p$
0-CELLS					
p_1		0	p_2		$\varphi(1/2) - \varphi(0) = q - p$
p_3		$\varphi(1) - \varphi(1/2) = p - q$	p_4		0

Before writing down the boundary maps, we make a final observation. In the cells $E, F, G,$ and H , the domain of f is as in figure 2 and the function f is constant on the hypotenuse $\{x = y\}$ of the underlying triangle. Hence for the purpose of writing

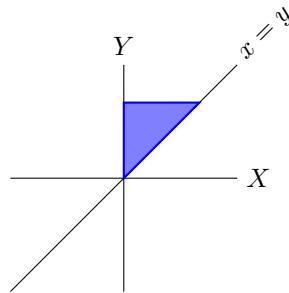


FIGURE 2. Solution space for case 2

down the boundary map, we can replace the cells with homeomorphic bigons whose boundary is given by 1–cells corresponding to the sides $\{x = 0\}$ and $\{y = 1/2\}$ of the initial triangles (and the third side is contracted to a point) [cf. figure 3].

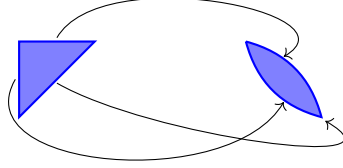


FIGURE 3. Homeomorphic solution space for case 2

By abuse of notation, we will denote these homeomorphic 2-cells by the same name. By symmetry, same thing is true for the four cells $I, J, K,$ and L . Then the boundary maps on the 2-cells are defined as follows.

$$\begin{aligned} \partial(A) &= \bar{\beta} - \alpha - \bar{\delta} + \gamma & \partial(B) &= \bar{\alpha} - \beta - \gamma + \bar{\delta} \\ \partial(C) &= \beta - \bar{\alpha} - \delta + \bar{\gamma} & \partial(D) &= \alpha - \bar{\beta} - \bar{\gamma} + \delta \\ \partial(E) &= \alpha + \beta & \partial(F) &= \beta + \alpha & \partial(G) &= \bar{\alpha} + \bar{\beta} & \partial(H) &= \bar{\beta} + \bar{\alpha} \\ \partial(I) &= \gamma + \delta & \partial(J) &= \delta + \gamma & \partial(K) &= \bar{\gamma} + \bar{\delta} & \partial(L) &= \bar{\delta} + \bar{\gamma} \end{aligned}$$

And finally the boundary maps from 1–cells to 0–cells are given by

$$\begin{aligned} \partial(\alpha) &= p_2 - p_1 & \partial(\beta) &= p_1 - p_2 & \partial(\gamma) &= p_3 - p_1 & \partial(\delta) &= p_1 - p_3 \\ \partial(\bar{\alpha}) &= p_3 - p_4 & \partial(\bar{\beta}) &= p_4 - p_3 & \partial(\bar{\gamma}) &= p_2 - p_4 & \partial(\bar{\delta}) &= p_4 - p_2 \end{aligned}$$

Refer to figure 4 to see how all the cells glue together with correct orientation.

3.2. Taking out some of the vertices of \mathcal{X} to get a homologically trivial 2-cycle. So far we have constructed a 2–dimensional complex, let's call it \mathcal{X} , with four 0–cells, eight 1–cells and twelve 2–cells. We can picture \mathcal{X} as in figure 4.

In last section, we defined a continuous vector-valued function from \mathcal{X} to \mathbb{R}^2 . Note that $f(p_1)$ and $f(p_4)$ are zero, however they correspond to the degenerate case when $\{p, q\} = \{r, s\}$. So to prove proposition 3.1, we need to show that f has a nontrivial zero on \mathcal{X} other than p_1 and p_4 .

Let us assume, for the sake of contradiction, that f is nonzero everywhere else on \mathcal{X} . We remove two circular ϵ -neighborhoods of p_1 and p_4 from \mathcal{X} and compactify (in the obvious way) the resulting space to get a new cell complex \mathcal{Y} . Clearly we have

$$\partial\mathcal{Y} = \text{Lk}(p_1, \mathcal{X}) \cup \text{Lk}(p_4, \mathcal{Y})$$

where $\text{Lk}(p_1, \mathcal{X})$ is the link of p_1 in \mathcal{X} . Now $f|_{\mathcal{Y}}$ is nonzero and so we can replace f by $f/\|f\|$, which by abuse of notation, we will still call f , to get a continuous function defined as

$$f : \mathcal{Y} \rightarrow S^1.$$

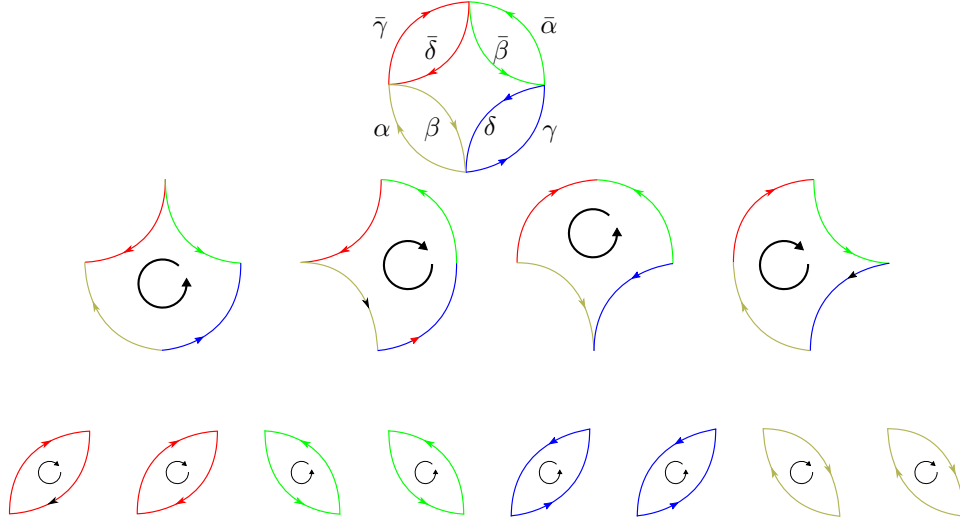


FIGURE 4. All cells with orientation

Then we have a commutative diagram

$$\begin{array}{ccc}
 \partial\mathcal{Y} & \xleftarrow{i} & \mathcal{Y} \\
 i \circ f \downarrow & & \swarrow f \\
 S^1 & &
 \end{array}$$

which induces following commutative diagram on the homology of these spaces

$$\begin{array}{ccc}
 H_1(\partial\mathcal{Y}) & \longrightarrow & H_1(\mathcal{Y}) \\
 \downarrow & & \swarrow \\
 H_1(S^1) = \mathbb{Z} & &
 \end{array}$$

Our goal is to show existence of cycles in $H_1(\partial\mathcal{Y})$ that are homologically trivial in $H_1(\mathcal{Y})$ but map nontrivially to $H_1(S^1)$. [For a short introduction to the theory of homology, the reader may refer to chapter 2 of [1].]

Recall that $\text{Lk}(p_1, \mathcal{X})$ (and similarly $\text{Lk}(p_4, \mathcal{X})$) is a graph whose vertices correspond to the 1-cells of \mathcal{X} which are incident to p_1 (resp. p_4). Two such edges are adjacent in $\text{Lk}(p_1, \mathcal{X})$ (resp. $\text{Lk}(p_4, \mathcal{X})$) if they are incident to a common 2-cell at p_1 (resp. p_4). From the list of cells above we can then easily see that both the links have 4 vertices and 8 edges (and the two graphs are disjoint from each other).

Now consider the cycle in $\text{Lk}(p_1, \mathcal{X})$ that consists of the 1-cell (edge in the graph) corresponding to E minus the 1-cell corresponding to F . Since the 2-cells clearly bound a sphere in \mathcal{X} , this cycle as an element of $H_1(\mathcal{Y})$ bounds a disk, and hence is homologically trivial. We will call this the $\alpha\beta$ -cycle because of the boundary of the 2-cells. Similarly we can construct $\gamma\delta$ -, $\bar{\alpha}\bar{\beta}$ - and $\bar{\gamma}\bar{\delta}$ -cycle in $H_1(\partial\mathcal{Y})$ which

map homologically trivially into $H_1(\mathcal{Y})$. In general we can produce such a cycle whenever a collection of 2-cells of \mathcal{X} makes a sphere.

3.3. Calculating degree of f on cycles of $\partial\mathcal{Y}$. We first concentrate on the ϵ -neighbourhood, let's call it U , of the point p_1 in \mathcal{X} that was used to obtained $\text{Lk}(p_1, \mathcal{X})$ as ∂U . Similarly, we denote the corresponding neighbourhood at p_4 by V . We want to choose U and V sufficiently small such that the following is true.

Suppose $U \cap \alpha = [0, a]$ and $U \cap \delta = [d, 1]$. We want the vectors $\vec{u}_\alpha = \varphi(a) - \varphi(0)$ and $\vec{u}_\delta = \varphi(1) - \varphi(d)$ approximately of same magnitude and in the same direction (note that $\varphi(0) = \varphi(1)$). Similarly if $U \cap \beta = [x, 1/2]$ and $U \cap \gamma = [1/2, c]$, then we want $\vec{u}_\beta = \varphi(1/2) - \varphi(b)$ and $\vec{u}_\gamma = \varphi(c) - \varphi(1/2)$ to be of approximately same magnitude and direction. Now f is differentiable in a neighbourhood of p_1 since φ is differentiable at p and q . So such a choice of U exists. Note that we can simultaneously choose V small enough such that the following are also satisfied.

$$\vec{u}_\alpha \approx \vec{u}_\delta, \vec{u}_\beta \approx \vec{u}_\gamma$$

and

$$\vec{u}_{\bar{\alpha}} = -\vec{u}_\alpha, \vec{u}_{\bar{\beta}} = -\vec{u}_\beta, \vec{u}_{\bar{\gamma}} = -\vec{u}_\gamma \text{ and } \vec{u}_{\bar{\delta}} = -\vec{u}_\delta.$$

Let θ denote the angle from the vectors \vec{u}_α to \vec{u}_β and θ' denote the angle from the vectors \vec{u}_γ to \vec{u}_δ (in clockwise direction, wlog). Since $\vec{u}_\alpha \approx \vec{u}_\delta$ and $\vec{u}_\beta \approx \vec{u}_\gamma$, one of the two angles θ and θ' is less than or equal to π . Without loss of generality let us assume that $\theta < \pi$. Then we know that the angle from \vec{u}_δ to \vec{u}_γ is also less than π .

Observe that the vectors u_α and u_β are essentially the direction of tangents to the knot K at 0 and $1/2$. Consequently since K is embedded, up to homotopy, the part of φ from 0 to $1/2$ looks like one of the following 3 possibilities in figure 5.

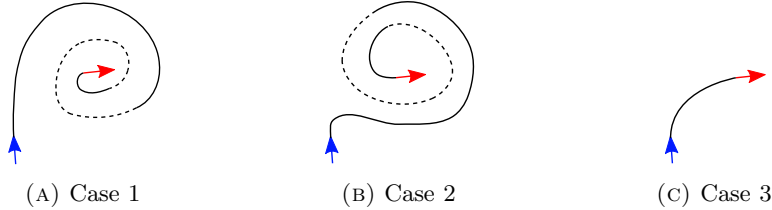


FIGURE 5. 3 Homotopy cases for part of K from 0 to $1/2$

Let $p_\alpha, p_\beta, p_\gamma$ and p_δ denote the vertices in $\text{Lk}(p_1, \mathcal{X})$ corresponding to the 1-cells α, β, γ and δ respectively (cf. figure 6).

We introduce the notation $[p_\alpha \rightarrow p_\beta]$ to denote the edge in from p_α to p_β in $\text{Lk}(p_1, \mathcal{X})$ corresponding to the 2-cell F (the orientation is derived from \mathcal{X} and the fact that $E - F$ is a sphere). Similarly we define $[p_\beta \rightarrow p_\alpha]$ to denote the edge corresponding to E . Thus the $\alpha\beta$ -cycle is equal to $[p_\alpha \rightarrow p_\beta] + [p_\beta \rightarrow p_\alpha]$.

We wish to calculate the degree of $f|_{\alpha\beta\text{-cycle}}$. Depending on the 3 cases for the embedding of $\varphi|_{[0, 1/2]}$ as discussed above, we will get different results.

Let's look at $f|_{[p_\beta \rightarrow p_\alpha]}$ first. Note that for small enough U i.e for $x \rightarrow 0+$ and $y \rightarrow 1/2-$, the function f on $[p_\beta \rightarrow p_\alpha]$ is approximately a linear interpolation from $f(p_\beta) = \vec{u}_\beta$ to $f(p_\alpha) = \vec{u}_\alpha$ and hence the image of f traverses the circle from the direction of \vec{u}_β to that of \vec{u}_α (cf. figure 7).

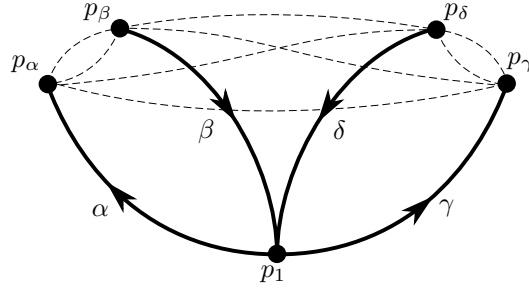


FIGURE 6. The link (dotted part) of the vertex p_1 in \mathcal{X}

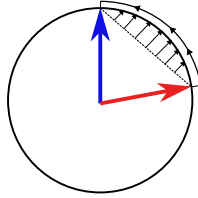


FIGURE 7. Image of f on $[p_\beta \rightarrow p_\alpha]$

The case of $[p_\alpha \rightarrow p_\beta]$ is very different because of the the corresponding 2-cell F . Note that by construction, a small neighbourhood of p_1 in F looks like

$$\{(x, y) \mid y - x \leq \epsilon\} \subset \{0 \leq x < y \leq 1/2\}$$

for some small $\epsilon > 0$ so that $\varphi(y) - \varphi(x) \approx 0$. So, as we traverse $[p_\alpha \rightarrow p_\beta]$, the function f maps continuously from $\varphi(\epsilon) - \varphi(0)$ to $\varphi(1/2) - \varphi(1/2 - \epsilon)$ through vectors of the form $\varphi(x + \epsilon) - \varphi(x)$ with x going from 0 to $1/2 - \epsilon$. Thus the winding number of image of f on $[p_\alpha \rightarrow p_\beta]$ depends on the embedding of φ from 0 to $1/2$. Looking at the 3 possible cases as in figure 5, we then observe that the image of f looks as in figure 8. Consequently the degree of f on the $\alpha\beta$ -cycle can be tabulated as in table 2.

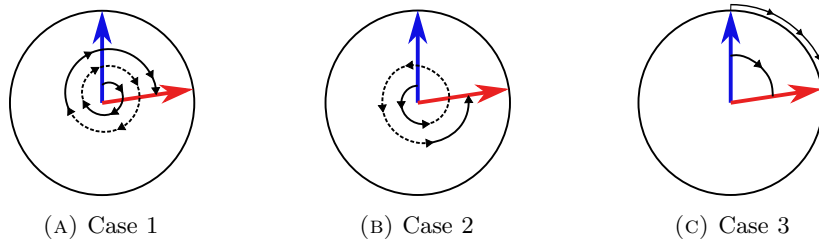
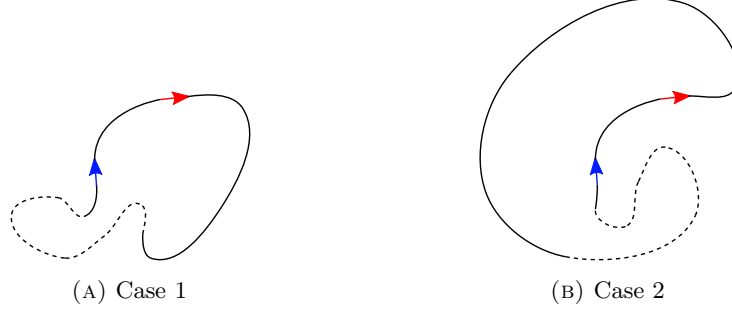


FIGURE 8. Image of f in the 3 cases

In the case when the degree of f on the $\alpha\beta$ -cycle is zero, we look at the possibilities for the part of the knot φ from $1/2$ to 1. Again up to homotopy, there are exactly 2 possible cases as in figure 9. In each case, let's calculate the degree of f on the $\gamma\delta$ -cycle which depends on the embedding of φ from $1/2$ to 1. Then it is easy to see that in both cases, $f|_{\gamma\delta\text{-cycle}}$ is non zero.

	Case 1	Case 2	Case 3
Degree of f on $\alpha\beta$ -cycle	≥ 0	≤ 0	$= 0$

TABLE 2

FIGURE 9. 2 Homotopy cases for part of K from $1/2$ to 1 for case 3 in figure 5

3.4. Conclusion. The proof of proposition 3.1 now follows from the following contradiction. Since the $\alpha\beta$ -cycle and the $\gamma\delta$ -cycle are homologically trivial in \mathcal{Y} , the composition

$$H_1(\partial\mathcal{Y}) \hookrightarrow H_1(W) \rightarrow H_1(S^1)$$

is a zero map. But on the other hand at least one of these cycle maps with nonzero degree to S^1 and hence maps nontrivially in homology. Thus our assumption in subsection 3.3 is wrong and f indeed must have a nontrivial zero other than p_1 and p_4 . \square

4. PROOF OF SALVATI'S THEOREM

Following Salvati, we first prove the following corollary to proposition 3.1 which after a easy induction argument will prove theorem 4.2.

Theorem 4.1. *Let $w_1, w_2 \in \{a, b, A, B\}^*$ such that $w_1 w_2 \in O_2$. Then there exists $x_1, x_2, y_1, y_2 \in \{a, b, A, B\}^*$ such that*

- (1) $w_1 w_2 \in \text{perm}(x_1 x_2 y_1 y_2)$,
- (2) $\max\{|x_1 x_2|, |y_1 y_2|\} < |w_1 w_2|$,
- (3) $w_1 w_2 \in \text{perm}\{x_1 x_2 y_1 y_2\}$, and
- (4) $x_1 x_2$ and $y_1 y_2$ are both in O_2 .

Proof. We consider the set of all possible choices of x_1, x_2, y_1 , and y_2 which satisfy condition (1), (2), and (3) from above. Let's call this set S . It is easy to see that S is nonempty and finite. We want to show that at least one of these choices will also satisfy condition (4).

Recall that each word in O_2 and in general any word in a, b, A, B corresponds to a path in the integer lattice in \mathbb{R}^2 . By abuse of notation, we will denote the path corresponding to w by w as well. Then let $\vec{v}(w)$ be the vector in \mathbb{R}^2 which goes from the starting point of w to the end point of w . We will use the notation (a, b)

to mean the point or the vector $\langle a, b \rangle = a\hat{i} + b\hat{j}$ interchangeably and the meaning should be clear from the context. Thus,

$$\vec{v}(w) = |w|_a \langle 1, 0 \rangle + |w|_b \langle 0, 1 \rangle + |w|_A \langle -1, 0 \rangle + |w|_B \langle 0, -1 \rangle$$

We will say a point (a, b) is an lattice point if both $a, b \in \mathbb{Z}$. Similarly we say a vector \vec{v} is an lattice vector if both its \hat{i} component and \hat{j} components are integer. Thus the vector (a, b) is an lattice vector if it is an lattice point.

For each choice in S , we observe that for (x_1, x_2, y_1, y_2) in S to satisfy (4) is equivalent to having $\vec{v}(x_1) + \vec{v}(x_2)$ equal to the zero vector i.e. the origin. Note that by construction, the image under P always gives an lattice vector.

We would like to apply proposition 3.1 to the closed lattice path corresponding to $w = w_1 w_2$, where the two fixed points p and q are the starting point of w_1 (i.e. the ending point of w_2 and also, the origin) and the starting point of w_2 (which is also the ending point of w_1). Unfortunately, the closed lattice path is clearly not differentiable at those two points. We circumvent this problem in the following way.

Firstly note that if either the starting or ending letter in w_1 is inverse of either the starting or ending letter in w_2 , then we can easily find x_1, x_2, x_3, x_4 as above. For example if $w_1 = aw'_1 b$ and $w_2 = bw'_2 A$, then $x_1 = a, x_2 = A, x_3 = w'_1 b$, and $x_4 = bw'_2$ will work. So we might as well assume that w_1 and w_2 do not have compatible letters at the starting or ending. This takes care of the condition that the tangents at p and q are not antiparallel.

Let's write $w_1 = s_1 w'_1 s_2$ and $w_2 = s_3 w'_2 s_4$, where $s_i \in \{a, b, A, B\}$ is a letter. Here we are assuming $|w_1|, |w_2| \geq 2$ since otherwise the proof is trivial. Now let's say $s_1 = a$. Then none of s_2, s_3 , and s_4 can be A . Also b and B both can not be in the set $\{s_2, s_3, s_4\}$. So we also assume $\{s_2, s_3, s_4\} = \{a, b\}$. The other cases are similar. Now the two possibilities for $s_4 s_1$, i.e. the two letters around the point p are aa or ab . For each of those possibilities $s_2 s_3$, the two letter around q can be one of the four

$$ab, aa, bb, ba$$

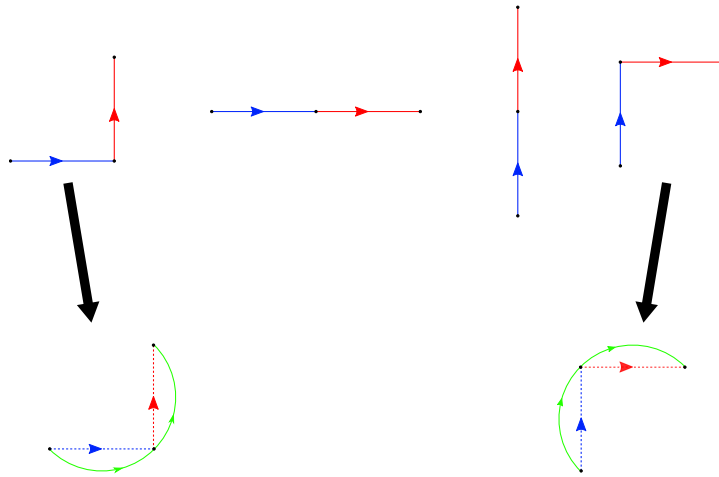


FIGURE 10. ‘Smooth’ening the path at p and q

For any of the above 8 possibilities, we replace the lattice path corresponding to ab and ba (only at p or q) by a smooth circular arc as in figure 10, so that smooth path still passes through the lattice points but none of the other points in the arc is a lattice point. If either of s_4s_1 and s_2s_3 is aa or bb then we leave it as it is. Thus we have changed our closed lattice path corresponding to w into another loop which is differentiable at p and q , and so we can now apply prop 3.1.

The proposition then gives us the existence of two points r and s which in turns gives us four vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and \vec{v}_4 , two of which add up to 0. It is easy to see that if r and s is actually a lattice point, then all four of these vectors are lattice vectors and we can easily use $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ suitably to get a choice of (x_1, x_2, y_1, y_2) in S which satisfies (4).

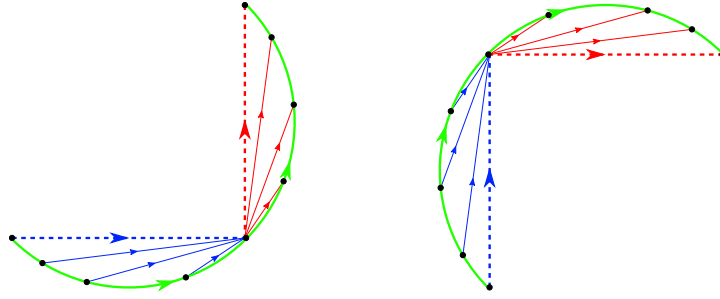


FIGURE 11. r or s cannot lie in the replaced part

Another interesting observation is that r and s cannot lie in the part of the lattice path we replaced in the previous paragraph to ‘smooth’en the loop at p and q . Because otherwise the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ would all lie in the same quadrant and cannot sum up to 0. (cf figure 11). So all we need to prove is that we can find points r and s satisfying proposition 3.1 which do not lie in the middle of a horizontal (corresponding to a or A) or a vertical (corresponding to b or B) segment in the modified closed lattice path.

Contrary to what we have to prove, suppose all choices of r and s satisfying the conclusion of proposition 3.1 have at least one of them a non-lattice point. We can assume wlog that r (resp. s) lies in the path corresponding to w_1 (resp. w_2). All the other possibilities can be dealt with very similar and easy arguments.

In the first case, for some choice of r and s as above, suppose r is a lattice point and s lies in the interior of a segment. Since p and q are lattice points by our construction, \vec{v}_1 and \vec{v}_2 are lattice vectors but \vec{v}_3 and \vec{v}_4 are not. Then the partition mentioned in proposition 3.1 must be $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_3, \vec{v}_4\}$ because otherwise they don’t add up to 0 (which is a lattice vector). But then we might as well choose s to coincide with q and then both r and s become lattice points and still satisfy prop 3.1. Contradiction.

Next suppose both r and s are not lattice points. Again if our choice for partition is $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_3, \vec{v}_4\}$ then we can replace r and s with p and q respectively and be done. Suppose r lies in the middle of a horizontal segment corresponding to a . Then the only way either $\vec{v}_1 + \vec{v}_3 = 0$ or $\vec{v}_1 + \vec{v}_4 = 0$ is if s also lies in the middle of a horizontal segment, since otherwise $\vec{v}_1 + \vec{v}_3$ (or $\vec{v}_1 + \vec{v}_4$) can not be a lattice point. Moreover, r and s both must divide the segments they belong to in equal or opposite ratios. Then we suitably replace r and s by the starting or end point

of the segments they are in, since doing so adds and subtracts same vector and so the sum remains unchanged (still equal to zero). This gives us a choice of lattice points r and s which also satisfy the conclusion of prop. 3.1. Contradiction.

Thus either way we can always find lattice points r and s and consequently a choice of $(x_1, x_2, y_1, y_2) \in S$ satisfying (4) in theorem 4.1. \square

Theorem 4.2. *If $w_1w_2 \in O_2$, then $\text{Inv}(w_1, w_2)$ is derivable and hence w_1w_2 will be in the language $L = \{w \mid S(w) \text{ is derivable in } G\}$ generated by G .*

Proof. We prove the theorem by inducting on the size of the word w_1w_2 . The base cases are trivial and follow from the rules in G . Theorem 4.1 and the third rule in G forms the induction step. \square

5. THE CASE OF 3 DIMENSION AND HIGHER

We would like to generalize above proof to the case of O_3 and conjecture the following.

Conjecture 5.1. *O_3 is a 3-MCFL and is generated by the following grammar. Consider*

$$G = (\Omega, \mathcal{A}, R, S)$$

where $\Omega = (\{S; \text{Circ}\}, \rho)$ with $\rho(S) = 1$ and $\rho(\text{Circ}) = 3$; $\mathcal{A} = \{a, A, b, B, c, C\}$ and R is made of rules that have one of the following forms:

- (1) $S(x_1x_2x_3) \leftarrow \text{Circ}(x_1, x_2, x_3)$
- (2) $\text{Circ}(t_1, t_2, t_3) \leftarrow \text{Circ}(x_1, x_2, x_3)$
where $t_1t_2t_3 \in \text{perm}(x_1x_2x_3aA) \cup \text{perm}(x_1x_2x_3bB) \cup \text{perm}(x_1x_2x_3cC)$
- (3) $\text{Circ}(t_1, t_2, t_3) \leftarrow \text{Circ}(x_1, x_2, x_3), \text{Circ}(y_1, y_2, y_3)$
where $t_1t_2t_3 \in \text{perm}(x_1x_2x_3y_1y_2y_3)$
- (4) $\text{Circ}(\epsilon, \epsilon, \epsilon)$

Computer experiments suggest that this is indeed true in the cases where the word length of some $w \in O_3$ is small enough. Note that the main ingredient of the proof in previous section is homology and degree theory which have direct analogues in case of 3 dimension and higher. As such there is an obvious algorithm to proceed with in those cases. However, it turns out that it's hard to come up with 3-cells corresponding to the 2-cells F and J in 3 dimension and unfortunately without those, all the relevant 2-cycles are null homologous. Nonetheless, we can make some interesting observations which might prove helpful.

Remark 5.2. Given $t_1t_2t_3 \in O_3$, computer experiments suggest that it is always possible to choose x_1, x_2, x_3, y_1, y_2 , and y_3 in such a way that $t_1t_2t_3 = x_1y_1x_2y_2x_3y_3$ and $x_1x_2x_3, y_1y_2y_3 \in O_3$; i.e. it is possible to break the word $t_1t_2t_3$ into six subwords such that the two sets of three *alternate* subwords together make two words of O_3 . This dramatically cuts down the number of cells we need to consider for finding a relevant 2-cycle.

Remark 5.3. Given $t_1t_2t_3 \in O_3$, computer experiments suggest that it is always possible to choose x_1, x_2, x_3, y_1, y_2 , and y_3 in such a way that $t_1t_2t_3 = x_1y_1x_2y_2x_3y_3$ and $x_1x_2x_3, y_1y_2y_3 \in O_3$; i.e. it is possible to break the word $t_1t_2t_3$ into six subwords such that the two sets of three *alternate* subwords together make two words of O_3 . This dramatically cuts down the number of cells we need to consider for finding a relevant 2-cycle.

REFERENCES

- [1] Allen Hatcher, *Algebraic Topology*, Cornell Univ. 3rd Ed., ISBN 0-521-79540-0
- [2] Mark-Jan Nederhof. *A short proof that O_2 is an MCFL*, <http://arxiv.org/abs/1603.03610>
- [3] Sylvain Salvati. *MIX is a 2-MCFL and the word problem in Z^2 is solved by a third-order collapsible pushdown automaton*. Journal of Computer and System Sciences, Elsevier, 2015, 81(7), pp.1252 - 1277.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS, 60637
E-mail address: subhadip@math.uchicago.edu